International Journal of Innovative Studies in Sciences and Engineering Technology (IJISSET)

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 6 Issue: 5 | 2020*

One approach to Matrix

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Abstract: *We have some special type of matrices. To explore the various kind of matrices, in my previous paper, we dealt about I-matrix,J-matrix, transprocal of matrix, transprocose of matrix, super orhoganality, etc…In this paper I come to speak about the existence of I-matrix, J-matrix, transprocal of matrix, transprocose of matrix.. Finally defining pro symmetric matrix, band matrix and hyper symmetric matrix which elements are in arithmetic progression and geometric progression. The entries in our matrices will come from some arbitrary, but fixed, field K.*

Keywords: *Exchange matrix or anti- diagonal matrix, transprocal and transprocose of matrices, reflect matrix, reflex identity, pro- symmetric, hyper matrix, band matrix and pro band matrix*

Introduction: In symmetric matrices, we have various types. In my previous paper I dealt some matrices and its propertiesAlso we diagonalize a certain non-singular matrix by I-operation and J-operation then prove, both ways we get same certain matrix. Furthermore we seek other ways for defining transprocal and transprocose matrix.

From [1]. **Thomas K. Huckle and Konrad Waldherr and Thomas Schulte-Herbruggen** - Exploiting Matrix Symmetries and Physical Symmetries in Matrix Product States and Tensor Trains, Technische Universit¨at M¨unchen, Boltzmannstr. 3, 85748 Garching, Germany, we know definitions of per symmetry, centro symmetry and their properties. From [2]. **D. STEVEN MACKEY, NILOUFER MACKEY, AND DANIEL M. DUNLAVY** - STRUCTURE PRESERVING ALGORITHMS FOR PERPLECTIC EIGENPROBLEMS Electronic Journal of Linear Algebra ISSN 1081-3810, A publication of the International Linear Algebra Society, Volume 13, pp. 10-39, February 2005, we know definition of Flip operation. From my previous paper [3], **Balasubramani Prema Rangasamy**-Matrix – one review, ALAMT-vol.9, no.3 (2019), we know the definitions of I-operaton, J-operation, Transprocal and Transprocose. Then from [4]. **Stephen Boyd**-Department of Electrical Engineering-Stanford University and **Lieven Vandenberghe** Department of Electrical and Computer Engineering, University of California, Los Angeles- Introduction to Applied Linear Algebra- Vectors, Matrices, and Least Squares, we know about some matrix operations. Finally also from [5]. **Wikipedia** we know the details of mathematicians and their works.

German mathematicians Otto Toeplitz introduced diagonal constant matrix and Hermann Hankel introduced anti diagonal constant matrix. In this paper I used Arithmetic progression and Geometric progression elements to construct diagonal constant (I-constant) matrices, anti-diagonal (J-constant) matrices, pro symmetric Hyper symmetric matrices and find inverse matrices for foresaid matrices like band matrix and pro band matrix.

For instance we see,

1. Pro symmetric matrix $:= \begin{bmatrix} a & b & c & d \\ b & b & c & d \\ c & c & c & d \\ d & d & d & d \end{bmatrix}$
2. Band matrix $:= \begin{bmatrix} a & b & 0 & 0 \\ b & f & c & 0 \\ 0 & c & f & d \\ 0 & 0 & d & e \end{bmatrix}$
3. Pro Band matrix $:= \begin{bmatrix} a & c & 0 & 0 & k \\ b & f & c & 0 & 0 \\ 0 & b & f & c & 0 \\ 0 & 0 & b & f & c \\ k & 0 & 0 & b & e \end{bmatrix}$
4. Hyper matrix $:= \begin{bmatrix} a & b & c & d & e \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b \\ e & d & c & b & a \end{bmatrix}$

Other definitions of Transprocal and Transprocose matrices:

1. **Transprocal matrix**: Let A be n x n matrix and J be an exchange matrix or J-identity matrix then i.e. $JAJ = A^{\neg}$.

Ex: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ and

$$J=\begin{bmatrix}0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 1 & 0 & 0\\ \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots\\ 1 & \cdots & 0 & 0 & 0\end{bmatrix} \text{ then}$$

$$JA=\begin{bmatrix}0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 1 & 0 & 0\\ \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots\\ 1 & \cdots & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{bmatrix}=\begin{bmatrix}a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\\ \vdots & \vdots & \vdots & .\cdot^{\cdot} & \vdots\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\end{bmatrix}$$

$$JAJ=\begin{bmatrix}a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\\ \vdots & \vdots & \vdots & .\cdot^{\cdot} & \vdots\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\end{bmatrix}\begin{bmatrix}0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 1 & 0 & 0\\ \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots\\ 1 & \cdots & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}a_{nn} & \cdots & a_{n3} & a_{n2} & a_{n1}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ a_{3n} & \cdots & a_{33} & a_{32} & a_{31}\\ a_{2n} & \cdots & a_{23} & a_{22} & a_{21}\\ a_{1n} & \cdots & a_{13} & a_{12} & a_{11}\end{bmatrix}$$

2. **Transprocose matrix or Flip matrix**: Let A be n x n matrix and J be an exchange matrix or j-identity matrix then

i.e. $JA^T J = A^{\triangleright} = A^F$.

Ex:

$$JA^T J=\begin{bmatrix}0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 1 & 0 & 0\\ \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots\\ 1 & \cdots & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a_{11} & a_{21} & a_{31} & \cdots & a_{n1}\\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2}\\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 1 & 0 & 0\\ \vdots & .\cdot^{\cdot} & \vdots & \vdots & \vdots\\ 1 & \cdots & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}a_{nn} & \cdots & a_{3n} & a_{2n} & a_{1n}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ a_{n3} & \cdots & a_{33} & a_{23} & a_{13}\\ a_{n2} & \cdots & a_{32} & a_{22} & a_{12}\\ a_{n1} & \cdots & a_{31} & a_{21} & a_{11}\end{bmatrix}$$

Reflect matrix

Definition 3: Let A, B be two square matrices. If matrix B is said to be reflex matrix of matrix A then it should satisfy AB = B = BA.

Ex:

$$\text{Let } A=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}, B=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix} \text{ then}$$

$$BA=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}-8+2+2 & 8-6-4 & 16-8-6\\ 8-3-1 & -8+9+2 & -16+12+3\\ -6+2+1 & 6-6+2 & 12-8-3\end{bmatrix}=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}=B$$

$$AB=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-8-8+12 & -4-6+8 & 4+2-4\\ 4+12-12 & 2+9-8 & -2-3+4\\ -4-8+9 & -2-6+6 & 2+2-3\end{bmatrix}=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}=B$$

.

Reflect idempotent

Definition 4: If a square matrix A is said to be reflect idempotent matrix, its mirror image (J-matrix) should satisfies $A_J{}^n = A_I$, for all positive even integer n and $A_J{}^n = A_J$ for all positive odd integer n.

Proof:

$$\text{Let } A=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix} \text{ then}$$

$$A_I=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix};\ A_J=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}.$$

$$A_J{}^2=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}16-8-6 & 8-6-4 & -8+2+2\\ -16+12+3 & -8+9+2 & 8-3-1\\ 12-8-3 & 6-6+2 & -6+2+1\end{bmatrix}=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}=A_I$$

$$A_J^3=A_J A_J^2=A_J A_I=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}-8+2+2 & 8-6-4 & 16-8-6\\ 8-3-1 & -8+9+2 & -16+12+3\\ -6+2+1 & 6-6+2 & 12-8-3\end{bmatrix}=\begin{bmatrix}-4 & -2 & 2\\ 4 & 3 & -1\\ -3 & -2 & 1\end{bmatrix}=A_J$$

By this way we get, for all positive even integer n, $A_J{}^n = A_I$ and for all positive odd integer n, $A_J{}^n = A_J$

Definition 5: If a square matrix A is said to be co-reflect idempotent matrix, in J-operation matrix A_I should satisfies $A_I{}^n = A_J$, for all positive even integer n and $A_I^{\,n} = A_I$ for all positive odd integer n.

Proof:

$$\left(A_I^{\,2}\right)_J = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -8+2+2 & 8-6-4 & 16-8-6 \\ 8-3-1 & -8+9+2 & -16+12+3 \\ -6+2+1 & 6-6+2 & 12-8-3 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 2 \\ 4 & 3 & -1 \\ -3 & -2 & 1 \end{bmatrix} = A_J$$

$$\left(A_I^{\,3}\right)_J = A_J A_I = \begin{bmatrix} -4 & -2 & 2 \\ 4 & 3 & -1 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 16-8-6 & 8-6-4 & -8+2+2 \\ -16+12+3 & -8+9+2 & 8-3-1 \\ 12-8-3 & 6-6+2 & -6+2+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A_I$$

By this way we get, for all positive even integer n in J-operation, such that $A_I^{\,n} = A_J$ and for all positive odd integer n in J-operation, such that $A_I^{\,n} = A_I$.

Definition 6: A square matrix A_I is said to be exchange inverse, if $[A_I A_J]_I = J$ or $[A_I A_J]_J = I$

Ex:

Let $A_J = \begin{bmatrix} 3 & 3 & 4 \\ -1 & 0 & -1 \\ -3 & -4 & -4 \end{bmatrix}$ and $A_I = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ then

$$A_I A_J = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 3 & 4 \\ -1 & 0 & -1 \\ -3 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 12-3-9 & 12+0-12 & 16-3-12 \\ -3+0+3 & -3+0+4 & -4+0+4 \\ -12+4+9 & -12+0+12 & -16+4+12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = J$$

$$A_I A_J = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 3 & 4 \\ -1 & 0 & -1 \\ -3 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 16-3-12 & 12+0-12 & 12-3-9 \\ -4+0+4 & -3+0+4 & -3+0+3 \\ -16+4+12 & -12+0+12 & -12+4+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Where A_I is the exchange inverse of A_J.

Diagonalization of certain matrix on I-operation and J-operation

Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ then

For I-Matrix,

$$|A - xI| = \begin{bmatrix} 1-x & -3 & 3 \\ 3 & -5-x & 3 \\ 6 & -6 & 4-x \end{bmatrix} = x^3 - 12x - 16 = 0.$$

So, $x_1 = -2; x_2 = -2; x_3 = 4$

Now, $P_I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $P_I^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 1 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$

then $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Thus,

$$P_I D P_I^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 1 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} = A$$

For J-Matrix,

$$|A - xJ| = \begin{bmatrix} 1 & -3 & 3-x \\ 3 & -5-x & 3 \\ 6-x & -6 & 4 \end{bmatrix} = x^3 - 4x^2 - 4x + 16 = 0.$$

So, $x_1 = -2; x_2 = 2; x_3 = 4$

Now, $P_J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $P_J^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-5}{4} \end{bmatrix}$

then $D = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Thus,

$$P_J^{-1} D P_J = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -\frac{1}{2} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} = A$$

From above we concluded, diagonalization of I-matrix gives I-diagonal and J-matrix gives J-diagonal.

Water image of the certain matrix

Definition7: If a matrix A is said to be A^W (water image of a certain matrix), it should be $A^W = JA$, where J is an exchange matrix or J-identity matrix.

Ex: General illustration of A^W :

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$ then

water image of certain matrix is

$$A^W = \begin{bmatrix} a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}.$$

Properties:

1. $(A^W)^W = A.$
2. $I^W \neq I.$
3. $(A \pm B)^W = A^W \pm B^W.$
4. $(kA)^W = kA^W.$
5. $(AB)^W \neq A^W B^W.$
6. $tr(JA) = tr(A^W)$
7. $\det(A) = -\det(A^W)$
8. $(A^{-1})^W = JA^{-1}$; more generally $(A^n)^W = JA^n, n \in Z$.
9. $(A^T)^W \neq (A^W)^T$

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ then

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix}$$

So, $A^{T^W} = \begin{bmatrix} a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \end{bmatrix}$

Now,

$A^W = \begin{bmatrix} a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}$ Then

$$A^{W^T} = \begin{bmatrix} a_{n1} & \cdots & a_{31} & a_{21} & a_{11} \\ a_{n2} & \cdots & a_{32} & a_{22} & a_{12} \\ a_{n3} & \cdots & a_{33} & a_{23} & a_{13} \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ a_{nn} & \cdots & a_{3n} & a_{2n} & a_{1n} \end{bmatrix}$$

Thus,

$(A^T)^W \neq (A^W)^T.$

Theorem1: A_J and A^W are transprocal to each other.

Proof:

Let A be a m x n matrix.

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$ then

$$A_J=\begin{bmatrix} a_{1n} & \cdots & a_{13} & a_{12} & a_{11} \\ a_{2n} & \cdots & a_{23} & a_{22} & a_{21} \\ a_{3n} & \cdots & a_{33} & a_{32} & a_{31} \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ a_{mn} & \cdots & a_{m3} & a_{m2} & a_{m1} \end{bmatrix}$$

And $A^W=\begin{bmatrix} a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}$

From the above matrices we concluded, A_J and A^W are transprocal to each other.

Pro symmetric matrix

Definition 8: Let A be an n x n matrix, its elements are in given form is called Pro-symmetric matrix.

$$A=\begin{pmatrix} a & b & c & . & . & . & \lambda \\ b & b & c & . & . & . & \lambda \\ c & c & c & . & . & . & \lambda \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & \lambda & \lambda & . & . & . & \lambda \end{pmatrix}$$

Band matrix

Definition 9: Let A be an n x n matrix, its elements are in given form is called *Band* matrix.

$$A=\begin{pmatrix} p & q & 0 & 0 & \cdots & 0 \\ q & r & q & 0 & \cdots & 0 \\ 0 & q & r & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q & r & q \\ 0 & 0 & \cdots & 0 & q & s \end{pmatrix}$$

Works:

1. Let A be an n x n pro symmetric matrix, its elements are in AP :

$$A=\begin{pmatrix} a & a+d & a+2d & . & . & . & b \\ a+d & a+d & a+2d & . & . & . & b \\ a+2d & a+2d & a+2d & . & . & . & b \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ b & b & b & . & . & . & b \end{pmatrix}.$$

Where $b=a+(n-1)d$ in which 'a' is a initial number and 'd' is difference between consecutive two numbers

Then

$$A^{-1}=\frac{(-1)^{n+1}}{bd^{n-1}}\begin{pmatrix} p & q & 0 & 0 & \cdots & 0 \\ q & r & q & 0 & \cdots & 0 \\ 0 & q & r & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q & r & q \\ 0 & 0 & \cdots & 0 & q & s \end{pmatrix}$$

Where

$p=(-1)^n bd^{n-2}\,;\, q=(-1)^{n+1}bd^{n-2}\,;$
$r=(-1)^n 2bd^{n-2}\,;\; s=(-1)^n(b-d)d^{n-2}$

2. Let A be an n x n pro symmetric matrix, its elements are in GP :

$$A=\begin{pmatrix} a & ar & ar^2 & . & . & . & b \\ ar & ar & ar^2 & . & . & . & b \\ ar^2 & ar^2 & ar^2 & . & . & . & b \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ b & b & b & . & . & . & b \end{pmatrix}$$

Where $b=ar^{n-1}$ and $a,b,r\in Z$.
Then

$$|A|=(-1)^{n+1}a^n r^{n(n-1)/2}(r-1)^{n-1};$$
$$A_{11}=(-1)^n a^{n-1}r^{n(n-1)/2}(r-1)^{n-2};$$

set $A_{11} \Rightarrow (-1)^{n+1} A_{12}$

Sub-diagonal, diagonal and super-diagonal elements are numbers, elsewhere 0.

$$A_{i(i+1)} = A_{(i+1)i}$$

$$A_{(i+1)(i+1)} = (-1)^n \left[A_{i(i+1)} + \frac{A_{i(i+1)}}{r} \right]$$; Assign

$\frac{A_{(i+1)i}}{r} \Rightarrow (-1)^{n+1} A_{(i+1)(i+2)}$.

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 \\ . & .. & . & . & . & . & . & . & . \\ . & . & . & . & . & . & .. & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & A_{(n-2)(n-3)} & A_{(n-2)(n-2)} & A_{(n-2)(n-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{(n-1)(n-2)} & A_{(n-1)(n-1)} & A_{(n-1)n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{n(n-1)} & A_{nn} \end{pmatrix}$$

From the above we concluded

Theorem 2: Inverse of Pro-symmetric matrix is band matrix and vice versa.

Hyper matrix

Definition10 : Let A be an n x n matrix then elements of matrix in following form is called *HYPER* matrix.

$$A = \begin{pmatrix} a & b & c & d & e & f \\ b & a & b & c & d & e \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ e & d & c & b & a & b \\ f & e & d & c & b & a \end{pmatrix}$$

Pro band matrix

Definition 11: Let A be an n x n matrix then elements of matrix in following form is called *Pro-band matrix.*

$$A = \begin{pmatrix} r & s & 0 & 0 & \cdots & l \\ s & t & s & 0 & \cdots & 0 \\ 0 & s & t & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s & t & s \\ l & 0 & \cdots & 0 & s & r \end{pmatrix}$$

1. Let

$$A = \begin{pmatrix} a & a+d & a+2d & . \; . \; . & b=a+(n-1)d \\ a+d & a & a+d & . \; . \; . & a+(n-2)d \\ a+2d & a+d & a & . \; . \; . & a+(n-3)d \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ b=a+(n-1)d & a+(n-2)d & a+(n-3)d & . \; . \; . & a \end{pmatrix}$$

Be a diagonal constant n x n matrix with arithmetic progression numbers. Upper diagonal and lower diagonal elements are same in same order. Then

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} r & s & 0 & 0 & \cdots & l \\ s & t & s & 0 & \cdots & 0 \\ 0 & s & t & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s & t & s \\ l & 0 & \cdots & 0 & s & r \end{pmatrix}$$

Where

$$|A| = (-1)^{n+1} 2^{n-2} d^{n-1} [2a + (n-1)d];$$

$$s = (-1)^{n+1} 2^{n-3} d^{n-2} [2a + (n-1)d];$$

$$l = (-1)^{n+1} 2^{n-3} d^{n-1};$$

$$r = (-1)^n (|s| - |l|);$$

Set $t = (-1)^n |2s|$. Other than l, r, s and t are 0.

2. Let

$$A = \begin{pmatrix} a & ar & ar^2 & . \; . \; . & ar^{n-1} \\ ar & a & ar & . \; . \; . & ar^{n-2} \\ ar^2 & ar & a & . \; . \; . & ar^{n-3} \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ ar^{n-1} & ar^{n-2} & ar^{n-3} & . \; . \; . & a \end{pmatrix}$$

Be a diagonal constant n x n matrix with geometric progression numbers. Upper diagonal and lower diagonal elements are same in same order. Then

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} p & s & 0 & 0 & \cdots & 0 \\ s & q & s & 0 & \cdots & 0 \\ 0 & s & q & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s & q & s \\ 0 & \cdots & 0 & 0 & s & p \end{pmatrix}$$

Where $|A| = (-1)^{n+1} a^n (r^2 - 1)^{n-1}$;

$s = (-1)^{n+1} a^{n-1} r(r^2 - 1)^{n-2}$;

$p = (-1)^n a^{n-1} (r^2 - 1)^{n-2}$;

$q = (-1)^n (r^2 + 1) a^{n-1} (r^2 - 1)^{n-2}$;

Other than p, q and s are 0.

From the above we concluded,

Theorem 3: Inverse of Hyper matrix with AP elements is Pro band matrix but inverse of Hyper matrix with GP elements is band matrix.

Models:

1. If pro symmetric matrix A is in anti-diagonal form

$$A = \begin{pmatrix} \lambda & . & . & . & c & b & a \\ \lambda & . & . & . & c & b & b \\ \lambda & . & . & . & c & c & c \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & \lambda & \lambda & . & . & . & \lambda \end{pmatrix} \text{ then}$$

$$A^{-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & q & s \\ 0 & \cdots & 0 & q & r & q \\ 0 & \cdots & q & r & q & 0 \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ q & r & q & 0 & 0 & 0 \\ p & q & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. If pro symmetric matrix A is in trans-diagonal form

$$A = \begin{pmatrix} \lambda & . & . & . & \lambda & \lambda & \lambda \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda & . & . & . & c & c & c \\ \lambda & . & . & . & c & b & b \\ \lambda & . & . & . & c & b & a \end{pmatrix} \text{ then}$$

$$A^{-1} = \begin{pmatrix} s & q & 0 & 0 & \cdots & 0 \\ q & r & q & 0 & \cdots & 0 \\ 0 & q & r & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q & r & q \\ 0 & 0 & \cdots & 0 & q & p \end{pmatrix}$$

3. If pro symmetric matrix A is in anti-trans-diagonal form

$$A = \begin{pmatrix} \lambda & . & . & . & \lambda & \lambda & \lambda \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ c & c & c & . & . & . & \lambda \\ b & b & c & . & . & . & \lambda \\ a & b & c & . & . & . & \lambda \end{pmatrix}$$

Then $A^{-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & q & p \\ 0 & \cdots & 0 & q & r & q \\ 0 & \cdots & q & r & q & 0 \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ q & r & q & 0 & 0 & 0 \\ s & q & 0 & 0 & 0 & 0 \end{pmatrix}$

4. If Hyper matrix A is in anti-diagonal form

$$A = \begin{pmatrix} f & e & d & c & b & a \\ e & d & c & b & a & b \\ d & c & b & a & b & c \\ c & b & a & b & c & d \\ b & a & b & c & d & e \\ a & b & c & d & e & f \end{pmatrix} \text{ then}$$

$$A^{-1} = \begin{pmatrix} l & 0 & \cdots & 0 & s & r \\ 0 & 0 & \cdots & s & t & s \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & s & t & s & 0 & 0 \\ s & t & s & \cdots & 0 & 0 \\ r & s & 0 & \cdots & 0 & l \end{pmatrix}$$

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