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Lebesgue and Outer Hausdorff Measures of Manifolds of Revolution

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Abstract. *The Lebesgue measure on open subsets of Euclidean space is unequivocally one of the most referenced measures used in multivariate integral analysis. This measure and other induced outer Hausdorff measures on manifolds with empty interiors are obtained via the standard metric upon integration of appropriate volume elements on these subsets of Euclidean space. This paper aims to elucidate the fluidity of computational tools for determining volumes in the case of manifolds with axial rotational symmetry. Certain standard tools from Riemannian geometry are employed for the illustration of fundamental techniques prerequisite for the precise evaluation, or tactful estimation, of multivariate integrals on manifolds embedded in Euclidean space.*

Keywords: *Euclidean Space, Standard Metric, Differential Forms, Multivariate Integration, Rotational Symmetry, Riemannian Manifolds, Induced Volume Elements.*

INTRODUCTION

Differential forms are the items integrated on manifolds in all multivariate integration applications, and the volume element is the fundamental differential form identified on every manifold. Open sets are the manifolds in Euclidean space with dimension coincident with that of the ambient space, meaning that open sets have the same volume element as the spaces containing them. Manifolds of dimension one less are called hypersurfaces of Euclidean space, with the difference in dimension implying a different volume element from that of the ambient Euclidean space. Hypersurfaces are highly didactic for purposes including vector algebra and their intrinsic geometric properties; purposes which we shall only reference at the basest levels in our study.

The only differential forms we may integrate over an n-dimensional manifold to obtain non-trivial results are differential n-forms on the manifold, of which the volume element is the prime example. Hence, given the canonical coordinate system $(x_i)_{i=1}^{m}$ on $\mathbb{R}^m$, a differential n-form (n ≤ m) can always be written as:

$$\sum_j f_j\,(x_i)_{i=1}^{m} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_n} ,$$

whereby the f_j's are smooth real-valued functions defined on the manifold. We identify "∧" as the wedge product operator which is antisymmetric and associative for differential one forms, such as the canonical projection maps dx_i. For other properties of the wedge product, the reader may refer to ([2], Page 14). For the relevance of differential forms in integration, we have the result:

$$\int_{M^n} f_j\,(x_i)_{i=1}^{m} dx_{i_1} dx_{i_2} \dots\ dx_{i_n} = \int_{M^n} f_j\,(x_i)_{i=1}^{m} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_n}$$

for any n-dimensional submanifoldM^n embedded in $\mathbb{R}^m$.

It is useful to hereby define the volume element for a given manifold. This differential form on $M^n$is defined as the n-form ω for which $\omega(e_1, e_2, \dots, e_n) = 1$ whenever $(e_1, e_2, \dots, e_n)$ is an orthonormal basis for the tangent space to M^n and $[e_1, e_2, \dots, e_n]$ is the usual orientation for the tangent space. (Our focus is only on orientable manifolds.) For open setsin $\mathbb{R}^m$, we identify the volume element to be $dx_1 \wedge dx_2 \wedge \dots \wedge dx_m$. For hypersurfaces, of which smooth boundaries of bounded open sets are prevalent examples, the volume form is identified as

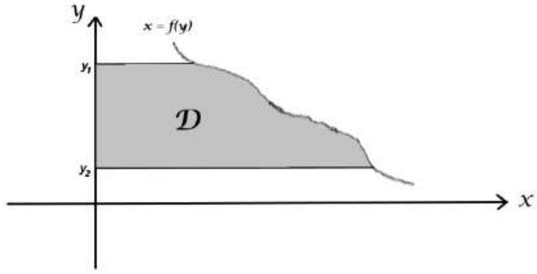
$$\sum_{j=1}^{m} (-1)^{j+1}\, n_j\, dx_1 \wedge dx_2 \wedge \dots \wedge \cancel{dx_j} \wedge \dots dx_m$$

whereby $(n_1, n_2, \dots, n_m)$ is the outward unit normal vector field or Gauss map of the manifold, and the strikethrough on any projection map in the given formula denotes its exclusion from the term.

We hereby reckon with the shorthand notations dV and dS to be respectively used for the volume elements of open sets of generic Euclidean space ($\mathbb{R}^m$) and hypersurfaces embedded in the space. As the name implies, integration of a volume form over a given manifold yields its volume – referred to as Lebesgue measure for open bounded sets and an induced outer Hausdorff measure for submanifolds of lower dimension. When precise results are required for multivariate integration procedures, axial symmetry of the target manifolds is a property that eases computational manipulations, when present. Otherwise, a fair grasp on applicable theoretical details can always facilitate the development of useful integral estimates.

With the aforementioned theoretical summary, we are now equipped to delve into the main intended content, which is scrutiny of methods for determining volumes of manifolds with rotational symmetry.

RESULTS



For the above sketch, the reader should take the region $\boldsymbol{D}$ as a two dimensional section of (x, y, z)-space $\mathbb{R}^3$, with the $z-$axis pointing outward from the page perpendicularly. Upon complete rotation of $\boldsymbol{D}$ by 2π radians about the y-axis in this sketch, we obtain a solid (open set) with y-axial rotational symmetry, which we shall denote U. Likewise, upon rotation of the graph for $[x = f(y)]$ by 2π radians about the y-axis in this sketch, we obtain a 2-dimensional surface with y-axial rotational symmetry, which we shall denote S. Formulae for computing the "volumes" of U and $S$are ubiquitously available in academic material published on methods of integral calculus in a single independent variable, except that we aim here to illustrate the efficacy of engagement of tools from differential forms in establishing them. To successfully implement this method, we require that unique tangent lines to the graph for $[x = f(y)]$exist at each point along the curve, meaning $f'(y)$must be well-defined on the interval (y_2, y_1). To initiate all computations with this approach, we first identify the matrix of rotation by θ radians about the y-axis, given as:

$$M = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$$

This matrix is compatible with the canonical rectangular coordinate system on $\mathbb{R}^3$.

Let us now investigate the volume of the solid of revolution. The depicted region $\boldsymbol{D}$iscomprised of points $(x, y, 0)$ such that $0 \leq x \leq f(y)$ and $y_2 \leq y \leq y_1$. We may apply the matrix M to the points of $\boldsymbol{D}$ to determine parametric rectangular equations (X, Y, Z) for all points in the solid manifold of revolution in terms of x, y and θ.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} . \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x.cos\theta \\ y \\ x.sin\theta \end{bmatrix}$$

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial \theta}d\theta = cos\theta dx - x.sin\theta d\theta$$

$$dY = \frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy + \frac{\partial Y}{\partial \theta}d\theta = dy$$

$$dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy + \frac{\partial Z}{\partial \theta}d\theta = sin\theta dx + x.cos\theta d\theta$$

$dX{\wedge}dY{\wedge}dZ$

$$= (cos\theta dx - x.sin\theta d\theta){\wedge}dy{\wedge}(sin\theta dx + x.cos\theta d\theta)$$

$$= (cos\theta dx{\wedge}dy - x.sin\theta d\theta{\wedge}dy){\wedge}(sin\theta dx + x.cos\theta d\theta)$$

$$= x.cos^2\theta dx{\wedge}dy{\wedge}d\theta - x.sin^2\theta d\theta{\wedge}dy{\wedge}dx$$

$$= x.cos^2\theta dx{\wedge}dy{\wedge}d\theta + x.sin^2\theta dx{\wedge}dy{\wedge}d\theta$$

$$= x\ dx{\wedge}dy{\wedge}d\theta.$$

The reader must recall the antisymmetric and associative properties of the wedge product between differential one-forms in order to come to terms with the above computational result. The antisymmetry in view makes it that $\alpha{\wedge}\alpha = -\alpha{\wedge}\alpha = 0$ for any differential one form α; of which the projection maps on Euclidean space are the prime examples.

The usual volume element on the open set of revolution U is the differential 3-form$dX{\wedge}dY{\wedge}dZ$ computed above, since (X, Y, Z) are the canonical rectangular coordinates for its points, determined via application of the rotation matrix M as shown above. We may now determine the required volume by employing straightforward multivariate integration techniques.

$$\text{Volume of } U = \int_U dX{\wedge}dY{\wedge}dZ = \int_0^{2\pi}\int_{y_2}^{y_1}\int_0^{f(y)} x\ dx{\wedge}dy{\wedge}d\theta = \int_0^{2\pi}\int_{y_2}^{y_1}\int_0^{f(y)} x\ dxdyd\theta$$

UsingFubini's theorem of multivariate integration to simplify the above expression;

$Volume\ of\ U$

$$= \int_0^{2\pi}\int_{y_2}^{y_1}\left[\frac{x^2}{2}\right]_{x=0}^{x=f(y)} dyd\theta$$

$$= 2\pi\int_{y_2}^{y_1}\left[\frac{\left(f(y)\right)^2}{2}\right]dy$$

$$= \int_{y_2}^{y_1}\pi(f(y))^2dy \qquad \rightarrow (\mathbf{1}).$$

We now investigate the "volume" (surface area) of S from rotation of the graph for $[x = f(y)]$ by 2π radians about the y-axis. The points along this graph can all be represented parametrically by $P(y) = (f(y), y, 0)$for $y_2 \leq y \leq y_1$. A tangent vector field along this curve obtained by differentiating with respect to the parameter y is $P'(y) = (f'(y), 1, 0)$. Hence, a normal vector field along this curve is $(1, -f'(y), 0)$, as it is clearly orthogonal (perpendicular) to $P'(y)$ within the 2-dimensional section depicted in the associated sketch above. As such, a unit normal vector field along the curve within this section is:

$$\vec{n} = \frac{(1, -f'(y), 0)}{\sqrt{1 + (f'(y))^2}}.$$

We must now apply the rotation matrix M given previously to points $P(y)$ in order to get rectangular parametric coordinates (X, Y, Z) for all points of S. We must also apply M to the vector field $\vec{n}$ to determine a unit normal vector field to the 2-dimensional manifold S.

$$M.P(y) = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} . \begin{bmatrix} f(y) \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} f(y).cos\theta \\ y \\ f(y).sin\theta \end{bmatrix} := \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$M.\vec{n} = \frac{1}{\sqrt{1 + (f'(y))^2}} \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} . \begin{bmatrix} 1 \\ -f'(y) \\ 0 \end{bmatrix} = \frac{1}{\sqrt{1 + (f'(y))^2}} \begin{bmatrix} cos\theta \\ -f'(y) \\ sin\theta \end{bmatrix} := \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$dX = \frac{\partial X}{\partial y} dy + \frac{\partial X}{\partial \theta} d\theta = f'(y).cos\theta dy - f(y).sin\theta d\theta$$

$$dY = \frac{\partial Y}{\partial y} dy + \frac{\partial Y}{\partial \theta} d\theta = dy$$

$$dZ = \frac{\partial Z}{\partial y} dy + \frac{\partial Z}{\partial \theta} d\theta = f'(y).sin\theta dy + f(y).cos\theta d\theta$$

The pertinent wedge products $dX \wedge dY, dX \wedge dZ$ and $dY \wedge dZ$ are thus computed.

$$dX \wedge dY = (f'(y).cos\theta dy - f(y).sin\theta d\theta) \wedge dy = f(y).sin\theta dy \wedge d\theta$$

$$dX \wedge dZ = (f'(y).cos\theta dy - f(y).sin\theta d\theta) \wedge (f'(y).sin\theta dy + f(y).cos\theta d\theta) = f'(y).f(y) dy \wedge d\theta$$

$$dY \wedge dZ = dy \wedge (f'(y).sin\theta dy + f(y).cos\theta d\theta) = f(y).cos\theta\ dy \wedge d\theta.$$

We hereby recall the volume element for S, which has been identified previously to be: $dS = N_1 dY \wedge dZ - N_2 dX \wedge dZ + N_3 dX \wedge dY$

$$= \frac{f(y)cos^2\theta + \left(f'(y)\right)^2 . f(y) + f(y)sin^2\theta}{\sqrt{1 + (f'(y))^2}} dy \wedge d\theta$$

$$= \frac{f(y)\left[1 + \left(f'(y)\right)^2\right]}{\sqrt{1 + (f'(y))^2}} dy \wedge d\theta$$

$$= f(y)\sqrt{1 + \left(f'(y)\right)^2} dy \wedge d\theta.$$

Hence, "Volume" of S is obtained as

$$\int_S dS = \int_0^{2\pi} \int_{y_2}^{y_1} f(y)\sqrt{1 + \left(f'(y)\right)^2} dy \wedge d\theta = \int_{y_2}^{y_1} 2\pi.f(y)\sqrt{1 + \left(f'(y)\right)^2} dy \rightarrow \mathbf{(2)}.$$

Worthy of note in the formula $(\mathbf{2})$derived above, the term $\sqrt{1 + \left(f'(y)\right)^2} dy$seen as part of the integrand is the element of arclength(ds)– a crucial differential 1-form required for solving any intrinsic integral along the curve $[x = f(y)]$. A great advantage of manifolds of revolution in the Euclidean 3-space (which is the ambient space of least dimension of interest) is that computation of their volumes reduces to a problem of evaluating line integrals, as opposed to other manifolds whose volumes have to be determined by evaluating a multiple integral after processing a comprehensive change of coordinate systems over the entire manifold via the standard Riemannian metric ([1], Page 62). This identified advantage extends to manifolds of revolution in higher dimensions, as we shall investigate in the succeeding conclusive section. The requirements for determining volumes of manifolds with axial rotational symmetry are volumes of appropriate balls and spheres, as compatible with the standard Riemannian metric.

CONCLUSION

The reader should again make reference to the diagram used in the previous section for the geometric analysis to be done here; extrapolating our results to higher dimensions. When rotating the section $\boldsymbol{D}$ by2π radians about the y-axis in $\mathbb{R}^m$, an open manifold of revolution would be obtained, given that the x-axis levels out a *hyperplane*containing all other axes (but the y-axis) from our limited perspective in the Euclidean space. Any hyperplane$[Y = y]$for $y_2 \leq y \leq y_1$ would therefore intersect the open manifold of revolution as a ball in this latterhyperplane with radius $f(y)$.The volume of the m-dimensional open manifold of revolution may be obtained by integrating the $(m-1)$ dimensional ball volumes for $y_2 \leq y \leq y_1$. We hereby arrive at the following proposition.

Proposition 1: The volume of the open set of revolution for the section $\boldsymbol{D}$by 2π radians about the y-axis in $\mathbb{R}^m$ is given as:

$$\int_{y_2}^{y_1} Vol[B^{m-1}(0, f(y))]dy \ ,$$

where$B^{m-1}(0, f(y))$ denotes the$(m-1)$ dimensional ball centered at the origin with radius $f(y)$. This proposition is immediately verified to corroborate the result (**1**)derived explicitly in the previous section. Moreover, this proposition is verified to validate the formulae for volumes of the unit balls in $\mathbb{R}^m (m \geq 3)$, wherein $f(y) = \sqrt{1-y^2} : -1 \leq y \leq 1$.

When rotating the curve $[x = f(y)]$bounding the 2-dimesional section ***D***about the y-axis in $\mathbb{R}^m$, a hypersurface of revolution would be obtained, given thatthe x-axis levels out ahyperplanecontaining all other axes (but the y-axis) from our limited perspective in the Euclidean space. Any hyperplane$[Y = y]$ for $y_2 \leq y \leq y_1$ would therefore intersect the hypersurface of revolution as an $(m-2)$ dimensional sphere in this latter hyperplane with radius $f(y)$. The volume of the $(m-$*1*dimensional hypersurface of revolution may be obtained by integrating the $(m-2)$ dimensional spherical volumes for $y_2 \leq y \leq y_1$. We hereby arrive at the following proposition.

Proposition 2: The volume of the hypersurface of revolution for the curve $[x = f(y)]$ bounding the 2-dimesional section ***D***by2π radians about the y-axis in $\mathbb{R}^m$is given as:

$$\int_{y_2}^{y_1} Vol[S^{m-2}(0, f(y))]ds \ ,$$

where$S^{m-2}(0, f(y))$ denotes the $(m-2)$ dimensional sphere centered at the origin with radius $f(y)$and $ds = \sqrt{1 + \left(f'(y)\right)^2}\, dy$ is the element of arclength on the differentiable curve.

This proposition is immediately verified to corroborate the result (**2**) derived explicitly in the previous section. Moreover, this proposition is verified to validate the formulae for volumes of the unit spheresin $\mathbb{R}^m (m \geq 3)$, wherein $f(y) = \sqrt{1-y^2} : -1 \leq y \leq 1$.

A variety of theorems for multivariate integration may be visited to wield theoretical and computational benefits of the above included differential-form approach for determining volumes of manifolds. For instance, we may visit Stokes' theorem for differentiable manifolds-with-boundary ([1], Page 172) to investigate the relationship between volumes of open sets and volumes of their boundaries. The relevance of the above findings in geometric integral analysis is self-evident. It is hoped that these findings would also be found relevant in a variety of other applied scientific research endeavors.

REFERENCES

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