International Journal of Innovative Studies in Sciences and Engineering Technology (IJISSET)

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 6 Issue: 1 | 2020*

The evolution equations of the Berwaldian curvatures

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Abstract: *Let (M,F) be a Berwaldian manifold. In this paper, the evolution equations of Ricci curvature and scalar curvature of (M,F) are given. As an application, the nonnegativity of the Ricci curvature under the Ricci flow is proved.*

Keywords: *Finslerian Berwaldian manifold, Ricci curvature, Scalar curvature, Ricci flow.*

1. INTRODUCTION

An evolutions equation of a metric, known as a Ricci flow, is very important in many fields of mathematics and physics [1]-[4]. In 1982, Hamilton [3] studied the evolution of the Riemannian curvatures under the Ricci flow.

The goal of this paper is to compute the evolution equations, under the Ricci flow, of the Ricci and scalar curvatures of a Finslerian Berwaldian manifold and find some of their applications. For this reason, every manifold is assumed to be connected and, any manifold and all mappings are supposed to be differentiable of classe C^∞.

The rest of this paper is organised as follows. In Section 2, we give some basic notions on Finslerian manifolds. The Section 3 is devoted to study the Berwaldian curvatures. In the Section 4, we derive the evolution equations of the Berwaldian Ricci and scalar curvature. As application we prove, in Section 5, the nonnegativity of the Berwaldian Ricci curvature.

2. RELIMINARIES

Let M be an n-dimensional manifold. We denote by T_xM the tangent space at $x\in M$ and by $TM:=\cup_{x\in M}T_xM$ the tangent bundle of M. Set $\mathring{T}M=TM\setminus\{0\}$ and π: $\mathring{T}M\to M$: $\pi(x,y)\to x$ the natural projection. Let $(x^1,\ldots,x^n)$ be a local coordinate on an open subset U of M and $(x^1,\ldots,x^n,y^1,\ldots,y^n)$ be the local coordinate on $\pi^{-1}(U)\subset TM$.

The local coordinate system $(x^i)_{\{i=1,\ldots,n\}}$ produces the coordinate bases

$$\{\frac{\partial}{\partial x^i}\}_{i=1,\ldots,n} \text{ and } \{dx^i\}_{i=1,\ldots,n}$$

respectively, for TM and cotangent bundle T^*M. We use Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise will be noted.

Definition 1. A function $F:TM\to M: [0,\infty)$ is called a Finslerian metric on M if :

(1) F is C^∞ on the entire slit tangent bundle $\mathring{T}M$,

(2) F is positively 1-homogeneous on the fibers of TM, that is
$\forall c>0, F(x,cy)=cF(x,y)$,

(3) the Hessian matrix $(g_{ij}(x,y))_{1\le i,j\le n}$ with elements

$$g_{ij}(x,y):=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^i\partial y^j} \tag{2.1}$$

is positive definite at every point (x,y) of $\mathring{T}M$.

Consider the differential map π_* of the submersion π: $\mathring{T}M\to M$. The vertical subspace of $T\mathring{T}M$ is defined by $V:=ker(\pi_*)$ and locally spanned by the set $\{F\frac{\partial}{\partial y^i}, 1\le i\le n\}$, on each $\pi^{-1}(U)\subset\mathring{T}M$.

An horizontal subspace H of $T\mathring{T}M$ is by definition any complementary to V. The bundles H and V give a smooth splitting

$$T\mathring{T}M=H\oplus V. \tag{2.2}$$

An Ehresmann connection is a selection of a horizontal subspace H of $T\mathring{T}M$. As explain in [5], H can be canonically defined from the geodesics equation.

Definition 2. Let $\pi:\mathring{T}M\to M$ be the restricted projection.

(1) An Ehresmann-Finsler connection of π is the subbundle H of $T\mathring{T}M$ given by

$$H := \ker\theta, \tag{2.3}$$

where $\theta: T\mathring{T}M \to \pi^*TM$ is the bundle morphism defined by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j). \tag{2.4}$$

(2) The form $\theta: T\mathring{T}M \to \pi^*TM$ induces a linear map

$$\theta|_{(x,y)}: T_{(x,y)}\mathring{T}M \to T_xM \tag{2.5}$$

for each point $(x,y) \in \mathring{T}M$; where $x = \pi(x, y)$.

The vertical lift of a section ξ of π^*TM is a unique section $v(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi^*(v(\xi))|_{(x,y)} = 0_{(x,y)} \text{ and } \theta(v(\xi))|_{(x,y)} = \xi_{(x,y)}. \tag{2.6}$$

(3) The form $\pi_*: T\mathring{T}M \to \pi^*TM$ induces a linear map

$$\pi_*|_{(x,y)}: T_{(x,y)}\mathring{T}M \to T_xM \tag{2.7}$$

for each point $(x,y) \in \mathring{T}M$; where $x = \pi(x, y)$.

The horizontal lift of a section ξ of π^*TM is a unique section $h(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi^*(h(\xi))|_{(x,y)} = \xi_{(x,y)} \text{ and } \theta(h(\xi))|_{(x,y)} = 0_{(x,y)}. \tag{2.8}$$

We have the following.

Definition 3. A Finslerian tensor field T of type $(q,0;p_1,p_2)$ on $\mathring{T}M$ is a C^∞ section of the tensor bundle

$$\underbrace{\pi^*T^*M \otimes \ldots \otimes \pi^*T^*M}_{p_1-times} \otimes \underbrace{T^*\mathring{T}M \otimes \ldots \otimes T^*\mathring{T}M}_{p_2-times} \otimes \bigotimes^q \pi^*TM. \tag{2.9}$$

Remark 1. In a local chart,

$$T = T^{k_1 \ldots k_q}_{i_1 \ldots i_{p_1} j_1 \ldots j_{p_2}} \partial_{k_1} \otimes \ldots \otimes \partial_{k_q} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_{p_2}}$$

where

$$(\partial_{k_1} \otimes \ldots \otimes \partial_{k_q} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_{p_2}})_{k \in \{1,\ldots,n\}^q, i \in \{1,\ldots,n\}^{p_1}, j \in \{1,\ldots,n\}^{p_2}}$$

is a basis section of this tensor and, the $\partial_{k_r} := \frac{\partial}{\partial x^{k_r}}$ as well as ε^{j_s} are respectively the basis sections for π^*TM and T^*T^0M dual of $T\mathring{T}M$.

Examples 1.

(1) The fundamental tensor g is of type $(0,0;2,0)$.

(2) The Ehresmann-Finsler form θ is of type $(1,0;0,1)$.

The following lemma defines the Chern connection on π^*TM.

Lemma 1. [5] Let (M,F) be a Finslerian manifold and g its fundamental tensor. There exists a unique linear connection ∇ on the vector bundle π^*TM such that, for all $X,Y \in \chi(\mathring{T}M)$ and for every $\xi,\eta \in (\pi^*TM)$, one has the following properties:

(i) $\nabla_X\pi_*Y - \nabla_Y\pi_*X = \pi_*[X,Y]$,

(ii) $X(g(\xi,\eta)) = g(\nabla_X\xi,\eta) + g(\xi,\nabla_X\eta) + 2A(\theta(X),\xi,\eta)$

where the tensor $\mathcal{A} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} dx^i \otimes dx^j \otimes dx^k$ is of Cartan. One has

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}, \; \Gamma^i_{jk} := \frac{1}{2} g^{il} \left(\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right) \tag{2.10}$$

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} = \mathbf{h}(\frac{\partial}{\partial x^i}) \right\}_{i=1,\ldots,n} \text{ with } N^i_j = \Gamma^i_{jk} y^k. \tag{2.11}$$

Definition 4. Let F be a Finslerian metric on an n-dimensional manifold M and $x \in M$. F is called a Berwald metric if, for a local coordinate $(x_i,y_i)_{i=1,\ldots,n}$ in $\mathring{T}M$, the Christoffel symbols Γ^l_{ij} of the Chern connection are only functions of the point x in M.

Example 2. All Riemannian metrics and locally Minkowskian metrics are examples of Berwald metrics. In fact,

(1) for Riemannian metrics,

$$\Gamma^i_{jk} = \gamma^i_{jk} = \tfrac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

In particular, the functions Γ^k_{ij} are independant of y.

(2) for locally Minkowskian metrics, in a neighborhood U of a point $x \in M$, the functions Γ^k_{ij} vanish identically. Hence, on U, Γ^k_{ij} can depend at most on x.

3. BERWALD RICCI AND SCALAR CURVATURES

Definition 5. The full curvature associated with the Chern connection ∇ on the vector bundle π^*TM over the manifold $\mathring{T}M$ is the application

$$\phi : \begin{array}{ccc} \chi(\mathring{T}M) \times \chi(\mathring{T}M) \times \Gamma(\pi^*TM) & \to & \Gamma(\pi^*TM) \\ (X,Y,\xi) & \mapsto & \phi(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \end{array}$$

By the relation (2.2), we have

$$\nabla_X = \nabla_{\hat{X}} + \nabla_{\check{X}},$$

where $X = \hat{X} + \check{X}$ with $\hat{X} \in \Gamma(H)$ and $\check{X} \in \Gamma(V)$. Using the metric F, one can define the full curvature of ∇ as:

$$\begin{aligned}\Phi(\xi,\eta,X,Y) &= g(\phi(X,Y)\xi,\eta)\\ &= g(\phi(\hat{X},\hat{Y})\xi+\phi(\hat{X},\check{Y})\xi+\phi(\check{X},\hat{Y})\xi+\phi(\check{X},\check{Y})\xi,\eta)\\ &= \mathbf{R}(\xi,\eta,X,Y)+\mathbf{P}(\xi,\eta,X,Y)+\mathbf{Q}(\xi,\eta,X,Y),\end{aligned}$$

where $\mathbf{R}(\xi,\eta,X,Y)=g(\varphi(\hat{X},\hat{Y})\xi,\eta)$, $\mathbf{P}(\xi,\eta,X,Y)=g(\varphi(\hat{X},\check{Y})\xi,\eta+g(\varphi(\check{X},\hat{Y})\xi,\eta)$ and $\mathbf{Q}(\xi,\eta,X,Y)=g(\varphi(\check{X},\check{Y})\xi,\eta)$ are respectively the first curvature, mixed curvature and vertical curvature. In particular, if ∇ is the Chern connection, the **Q-**curvature vanishes. In a local coordinate system, the components of the Chern curvature are:

$$\begin{aligned}\Phi(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l) &= \mathbf{R}(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l)+\mathbf{P}(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l)\\ &= \Big(\frac{\delta\Gamma^s_{il}}{\delta x^k}-\frac{\delta\Gamma^s_{ik}}{\delta x^l}\Big)g_{js}+\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr}-F\frac{\partial\Gamma^s_{ik}}{\partial y^l}g_{js}.\end{aligned} \tag{3.1}$$

Remark 2. In local coordinate, the curvatures **R** and **P** can also be found in [6].

If *F* is a Berwaldian metric then, by the Definition 4, the curvature associated with the Chern connection is

$$\Phi_{ijkl} = \Big(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\Big)g_{js}+\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr} \tag{3.2}$$

where $\Phi_{ijkl}=\Phi(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l)$.

With respect to the Chern connection, we have the following.

Definition 6.

(1) The Berwaldian Ricci tensor Ric of *(M,F)* is defined by

$$\mathbf{Ric}(\xi,X) := trace_g\Big[\eta\longmapsto R\big(X,\mathbf{h}(\eta)+\mathbf{v}(\eta)\big)\xi\Big]. \tag{3.3}$$

(2) The Berwaldian scalar curvature *Scal* of *(M,F)* is defined by

$$\mathbf{Scal} := trace_{\underline{g}}\Big(\mathbf{Ric}\Big),\ \underline{g}:=\pi^*g. \tag{3.4}$$

Locally, we have

$$\mathbf{Ric}(\partial_i,\hat{\partial}_k+\check{\partial}_k) = \frac{\partial\Gamma^l_{il}}{\partial x^k}-\frac{\partial\Gamma^l_{ik}}{\partial x^l}+\Gamma^s_{ik}\Gamma^l_{ls}-\Gamma^s_{il}\Gamma^l_{ks} \tag{3.5}$$

and

$$\mathbf{Scal} = \Big(\frac{\partial\Gamma^l_{il}}{\partial x^k}-\frac{\partial\Gamma^l_{ik}}{\partial x^l}+\Gamma^s_{ik}\Gamma^l_{ls}-\Gamma^s_{il}\Gamma^l_{ks}\Big)g^{ik}. \tag{3.6}$$

4. THE EVOLUTION EQUATIONS OF THE BERWALD RICCI AND SCALAR CURVATURE

Consider a manifold *M*, a one-parameter family $\{F_t\}_{t\in[0,\tau)}$ of Finslerian metrics on *M* of scalar flag curvature [7] and $\{g_t\}_{t\in[0,\tau)}$ its associated family of fundamental tensors. We call the Finslerian horizontal Ricci deformation the evolution equation on (M,F_t) given by

$$\frac{\partial}{\partial t}\underline{g}_t := -2\mathbf{Ric}_{F_t} \text{ with } F_t|_0 = F \tag{4.1}$$

where $\underline{g}_t$ is the pullback of g_t by the submersion $\pi: \mathring{T}M\to M$.

The existence of solutions of (4.1) is known in special cases, particulary in Berwald spaces, [1].

We obtain the following.

Lemma 2. Under the Ricci flow of a Berwaldian manifold *(M,F)*, the Christoffel symbols Γ^k_{ij} satisfies the following evolution equation

$$\begin{aligned}\frac{\partial\Gamma^k_{ij}}{\partial t} &= g^{kl}\big(-\nabla_i\boldsymbol{Ric}_{jl}-\nabla_j\boldsymbol{Ric}_{il}+\nabla_l\boldsymbol{Ric}_{ij}\big)\\ &\quad+2g^{sk}g^{rl}\boldsymbol{Ric}_{rs}\big(\nabla_i g_{jl}+\nabla_j g_{il}-\nabla_l g_{ij}\big).\end{aligned} \tag{4.2}$$

Proof. The Lemma 2 is obtained by using the equation (4.1) and the fact that

$$\begin{aligned}\frac{\partial g^{kl}}{\partial t} &= -g^{sk}g^{rl}\frac{\partial g_{rs}}{\partial t}\\ &= 2g^{sk}g^{rl}\mathbf{Ric}_{rs}.\end{aligned} \tag{4.3}$$

□

In the sequel, we use the Berwaldian Bianchi identity given in the

Lemma 3. If $\xi,\eta\in\Gamma(\pi*TM)$ and $X,Y,Z\in\chi(\mathring{T}M)$ then

$$(\nabla_Z\Phi)(\xi,\eta,X,Y)+(\nabla_X\Phi)(\xi,\eta,Y,Z)+(\nabla_Y\Phi)(\xi,\eta,Z,X)=0. \tag{4.4}$$

Proof. The Lemma 3 is obtained from the symmetry of ∇ and the Jacobi identity.

□

By contracting twice on equation (4.4) written in a local coordinate, we have

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 6 Issue: 1 | 2020*

$$\frac{1}{2}\nabla_j \mathbf{Scal} = \nabla^i \mathbf{Ric}(\partial_i, \hat{\partial}_j). \tag{4.5}$$

Using this last relation and by the fact that $Ric_{ik}:=g_{jl}\Phi_{ijkl}$ we get

$$\frac{\partial \mathbf{Ric}_{ik}}{\partial t} = \frac{\partial g^{jl}}{\partial t}\Phi_{ijkl} + g^{jl}\frac{\partial \Phi_{ijkl}}{\partial t}. \tag{4.6}$$

It follows, from the Lemmas 2 and 3 and the equation (4.6), that

Theorem 1. Under the Ricci flow of a Berwaldian manifold *(M,F)*, the Ricci tensor *Ric* satisfies the following evolution equation

$$\frac{\partial \boldsymbol{Ric}_{ik}}{\partial t} = \Delta \boldsymbol{Ric}_{ik} - 2\boldsymbol{Ric}^{jl}\Phi_{ijkl} - 2g^{jl}\boldsymbol{Ric}_{ij}\boldsymbol{Ric}_{kl} + B_{ik} \tag{4.7}$$

where

$$\begin{aligned}
B_{ik} = {} & 2\boldsymbol{Ric}_{pq}\Big(\nabla_l g_{ir} + \nabla_i g_{lr} - \nabla_r g_{il}\Big)\Big(\nabla_k g^{lp} g^{qr} + g^{lp}\nabla_k g^{qr}\Big) \\
& -2\boldsymbol{Ric}_{pq}\Big(\nabla_k g_{ir} + \nabla_i g_{kr} - \nabla_r g_{ik}\Big)\Big(\nabla_l g^{lp} g^{qr} + g^{lp}\nabla_l g^{qr}\Big) \\
& +2g^{lp}g^{qr}\Big[\nabla_k \boldsymbol{Ric}_{pq}\Big(\nabla_l g_{ir} + \nabla_i g_{lr} - \nabla_r g_{il}\Big) \\
& +\boldsymbol{Ric}_{pq}\Big(\nabla_k\nabla_l g_{ir} + \nabla_k\nabla_i g_{lr} - \nabla_k\nabla_r g_{il}\Big)\Big] \\
& -2g^{lp}g^{qr}\Big[\nabla_l \boldsymbol{Ric}_{pq}\Big(\nabla_k g_{ir} + \nabla_i g_{kr} - \nabla_r g_{ik}\Big) \\
& +\boldsymbol{Ric}_{pq}\Big(\nabla_l\nabla_k g_{ir} + \nabla_l\nabla_i g_{kr} - \nabla_l\nabla_r g_{ik}\Big)\Big] \\
& -\nabla_k g^{lr}\Big(\nabla_l \boldsymbol{Ric}_{ir} + \nabla_i \boldsymbol{Ric}_{il} - \nabla_r \boldsymbol{Ric}_{il}\Big) \\
& +\nabla_l g^{lr}\Big(\nabla_k \boldsymbol{Ric}_{ir} + \nabla_i \boldsymbol{Ric}_{ik} - \nabla_r \boldsymbol{Ric}_{ik}\Big) \\
& -g^{lr}\Big(\nabla_k\nabla_l \boldsymbol{Ric}_{ir} + \nabla_k\nabla_i \boldsymbol{Ric}_{il} - \nabla_k\nabla_r \boldsymbol{Ric}_{il}\Big) \\
& +g^{lr}\Big(\nabla_l\nabla_i \boldsymbol{Ric}_{ik} - \nabla_l\nabla_r \boldsymbol{Ric}_{ik}\Big).
\end{aligned} \tag{4.8}$$

with $\nabla_i = \frac{\delta}{\delta x^i}$ *and* $\Delta = g^{jl}\frac{\delta^2}{\delta x^j \delta x^l}$.

Now, with respect to the Berwaldian scalar curvature we have

$$\begin{aligned}
\frac{\partial \mathbf{Scal}}{\partial t} &= \frac{\partial g^{ik}}{\partial t}\mathbf{Ric}_{ik} + g^{ik}\frac{\partial \mathbf{Ric}_{ik}}{\partial t} \\
&= \Big(2g^{sk}g^{ri}\frac{\partial g_{rs}}{\partial t}\Big)\mathbf{Ric}_{ik} + g^{ik}\frac{\partial \mathbf{Ric}_{ik}}{\partial t}.
\end{aligned} \tag{4.9}$$

Using the relation (4.3) and the Theorem 1, we obtain

$$\begin{aligned}
\frac{\partial \mathbf{Scal}}{\partial t} &= \Big(2g^{si}g^{rk}\mathbf{Ric}_{ik}\Big) \\
&\quad +g^{ik}\Big(\Delta \mathbf{Ric}_{ik} - 2\mathbf{Ric}^{jl}\Phi_{ijkl} - 2g^{jl}\mathbf{Ric}_{ij}\mathbf{Ric}_{kl} + B_{ik}\Big) \\
&= 2g^{si}g^{rk}\mathbf{Ric}_{ik} + \Delta \mathbf{Scal} + G^{ik}B_{ik}.
\end{aligned} \tag{4.10}$$

Hence, we have the following:

Theorem 2. Under the Ricci flow of a Berwaldian manifold *(M,F)*, the scalar tensor *Scal* satisfies the following evolution equation

$$\frac{\partial \boldsymbol{Scal}_{F_t}}{\partial t} = 2g^{si}g^{rk}\boldsymbol{Ric}_{ik} + \Delta \boldsymbol{Scal} + g^{ik}B_{ik}. \tag{4.11}$$

5. APPLICATION: NONNEGATIVITY OF THE BERWALDIAN RICCI CURVATURE

We use the following result given in [3], which generalizes the maximum principle to tensors. Let u^i be a vector field, g_{ik} , M_{ik} and N_{ik} be symmetric tensors on a compact manifold X which may all depend on time t. Suppose that $N_{ik}=p(M_{ik},g_{ik})$ is a polynomial in M_{ik} formed by contracting products of M_{ik} with itself using the metric. Require that this polynomial satisfies the condition that whenever v^i is a null-eigenvector of M_{ik} so that $M_{ik}v^i=0$ for all k then one has $N^{ik}v^iv^i>0$. Then the following result is proved.

Lemma 4. [3] Suppose that on $0<t<\tau$

$$\frac{\partial}{\partial t}M_{ik} = \Delta M_{ik} + u^i\partial_j M_{ik} + N_{ik}, \tag{5.1}$$

where $N_{ik}=p(M_{ik},g_{ik})$ satisfies the null-eigenvector condition above. If $M_{ik}>0$ at $t = 0$, then it remains so on $0<t<\tau$.

Now, we have the following.

Theorem 3. Suppose that the evolution equation (4.7) has a solution on the interval $0<t<\tau$. If $Ric_{ik}\geq 0$ at $t=0$ then $Ric_{ik}\geq 0$ on $0<t<\tau$.

Proof. The Theorem 3 follows by using the Lemma 4 with $u^k=0$, $M_{ik}=Ric_{ik}$ and $N_{ik}=-2Ric_{jl}\Phi_{ijkl}-2g_{jl}Ric_{ij}Ric_{kl}+B_{ik}$. □

6. CONCLUSION

With the main results of this work, we are studying the evolution equation of the Finslerian Einstein curvature and its applications.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest regarding the publication of this paper.

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