Application of the Renormalization Group Theory to analyze the surface impedance of multifractal middle third Cantor set

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Abstract: *In this paper, we present a combination between the Renormalization Group Theory (RGT) and the MS-GEC to characterize the surface impedance of a multifractal Cantor set at a high fractalization stage. In fact, due to their scale invariance and the non-integer fractal dimensions, multifractal shapes are classified as a critical phenomenon, and they are very adapted to the Renormalization Group Theory mechanism. When applying the Multiscale-Generalized Equivalent Circuit (MS-GEC) method in its current form to analyze thesurface impedance of multi-fractal planar structure at infinity fractalization level leads to huge matrix size which block the solution, in addition to an increasing code volume and more CPU and memories resources must be provided with loss of accuracy.*

To overcome these difficulties and to save computing resources, a combination between the MS-GEC and the Renormalization Group Theory RGT is applied to characterize the surface impedance of the multifractal Cantor set. The idea is: knowing the renormalization function and its fixed point we can deduce the surface impedance of our multifractal structure at the infinity of the fractalization stage without the need to the detailed electromagnetic computation.

Keywords: *Renormalization Group Theory, Multifractal geometry, Electromagnetic Diffraction, surface impedance.*

1. INTRODUCTION

Fractal and multi-fractal geometry has inspired many researchers in there working field. Since it has been mathematically founded by Mandelbrot and Falconer [1], such structure has been studied and applied in many scientific fields like medicine [2], [3], biology [4] image and signal processing [5] as well as in antennas and electromagnetic in which we are interested in .In fact, designing a Cantor fractal printed antenna has improved the antennas performances: bandwidth and gain, for ISM band application [6]. Comparing to the rectangular patch antennas with its tow dimension width and length the fractal antenna has more geometrical parameter that gives more flexibility to control the antenna performances in addition to the diversity of fractal geometry as the Koch like antennas[7][8][9][10] which cover the WAN LAN WIFI bands.

Also fractal structure has been deployed in RFID tag [11] where a modified multi-fractal Cantor chipless RFID to enhance the read range was designed , in addition to the IoT application through a frequency reconfigurable Fractal circular patch antennas combined to diode pin [12].

In front of the increasing deployment of fractal and multifractal structures in RF applications a deep understanding of their electromagnetic behavior becomes a necessity. Hence the study of their surface impedance were also attractive for many researchers : in [13], the fractal Cantor planar structure is considered as a planar discontinuity put transversally inside a wave guide. The Multi-Scale Generalized Equivalent Circuit [14] method (MS-GEC) is used to compute its surface impedance through the diffraction operator $\hat{\Gamma}$. Splitting the fractal structure to an elementary structure and replacing it by their impedance operator $\widehat{Z_s}$in the next fractalization stage, the impedance operator $\widehat{Z_s}$ is constructed by selecting the active modes which are the strongly coupled modes between two adjacent elementary substructures applying this formulation at higher iteration level of the fractal became hard and needs huge computing resources. And the MS-GEC presents a limitation when fractalization tends to infinity.

To overcome this limitation, we focus on an interesting feature of fractals which is their self-similarity or scaling invariance which let them classified as a critical phenomenon. In fact the dimension is non integer and they present an exponent anomaly letting fractal measurements very adequate to the Renormalization Group theory mechanism which was first introduced in Quantum Filed and phase transition studies.

This property of fractals is also too interesting in studying their electromagnetic wave diffraction through the application of the Renormalization Group Theory on a fluctuation like the surface impedance Z_s verifying:

$$\vec{E} = Z_s(\vec{n} \times \vec{H})$$

We organize our work as follow: first, we present the multifractal planar geometry generated by Iterated Function System (IFS), precisely the multifractal Cantor set. In second part we introduce the Multiscale Generalized Equivalent Circuit Method: MS-GEC. In the third part we present the Renormalization Group Theory (RGT) concept and its steps. And finally, we apply in steps the surface impedance model formulation combined to the RGT to compute the surface impedance for Cantor set multifractal planar structure at the infinite.

2. CONSTRUCTING MULTIFRACTAL GEOMETRY BY ITERATED SYSTEM FUNCTION

The most practical and useful technique for generating fractal or multi-fractal geometry is the use of similar contractors, known as the Iterated System Functions: (ISF).As defined by Falconer in [15] a similar contractor is a linear function S defined by $|S_i(x) - S_i(y)| = c_i|x - y|$ where $0 < c_i < 1$ and x, y are two points in space. A finite family of contractors S_i: $\{S_1, S_2, \dots, S_m\}$ is named an Iterated System Function which operates recursively on the generator and leads to a complex fractal or multifractal geometry. On figure 1 we show the two contractors S_1 and S_2 which operates recursively on the generator E for two fractalization levels.

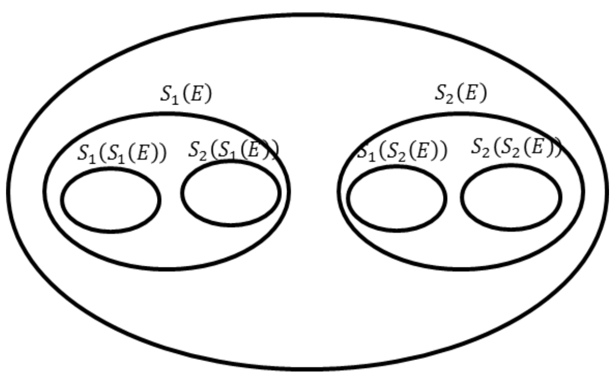

Fig -1: Iterated Function System scheme

2.1 Properties of fractals

The IFS constructed fractal or multifractal are characterized by their self-similarities: from scale to scale the fractal is composed of scaled copies of itself. In another word: a given fluctuation defined on scale s_1 can be also deduced from those on the scale s_2.

Their invariant set, named also attractor, is defined by

$$F = \bigcup_{i=1}^{m} S_i(F)$$

which is very interesting when applying the renormalization group theory, it is the fixed set of the fractalization and the fluctuation also converge to fixed value.

Mathematically the fixed set is defined as: if $S(E) = \cup_{i=1}^{m} S_i(E)$ S has a unique fixed set that verify $S(F) = F$ moreover $S^k(E) \to F$ when k→ ∞ , $S^k(E) = S(S^{k-1}(E))$

Non integer Dimensions:

As defined in [16] the Hausdroff dimension of fractal geometry defined by IFS is 's' : solution of the equation below:

$$\sum_{i=1}^{m} c_i^s = 1$$

The solution can be single number and the geometry is named fractal with its fractal dimension s or it can be also more than one number and the geometry is qualified by multifractal and it has multi fractal dimension.

2.2 The Multifractal Cantor set

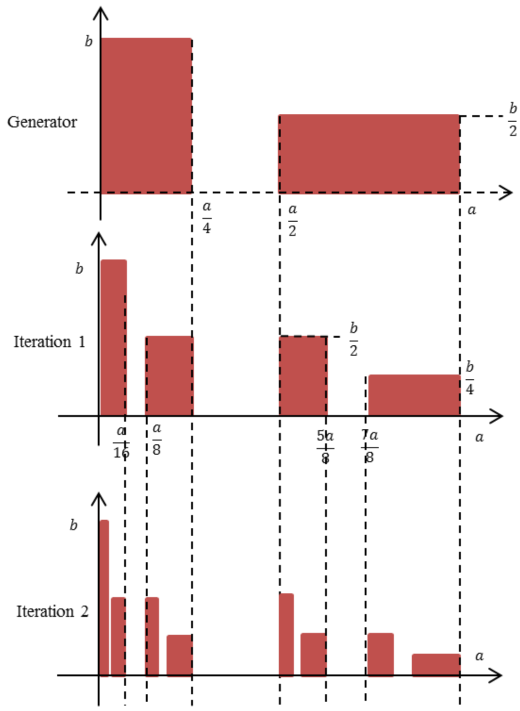

Fig -2: Multifractal Cantor set

The figure 2 represents the generator, the iteration 1 and iteration 2 of the multifractal Cantor set. The same process applied previously on fractal can be done with different scale of the contractors: instead using the same contraction in $f_1$and f_2 we adopt a different scaling for both x and y axis since it is a planar shape.

$$S = \begin{cases} f_1(x, y) = \left(\frac{1}{4}x, y\right) \\ f_2(x, y) = \left(\frac{1}{2}x + \frac{a}{2}, \frac{1}{2}y\right) \end{cases}$$

The use of Iterated System Function defines the fractal or the multifractal in unique way as described in []. In fact having the IFS and the generator is sufficient to build the fractals and they act as a key generation of the fractal. It was used for chipless RFID tag to give them more security in their geometry [17]. This gives more security to avoid malware reproduction

Applying the MS-GEC to compute the surface impedance of fractal or multifractal planar structure at high iteration level is a time consuming and it needs an important computing resources. And it became impossible when fractalization goes to infinity.

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To overcome these difficulties we combine the renormalization group theory with MS-GEC to compute the surface impedance of multifractal structure.

The fractal shape are a particular case of the multifractal ones so our study will be done one a multifractal structure and we can deduce the surface impedance for fractal one.

3. RENORMALIZATION GROUP METHOD COMBINED TO MS-GEC SURFACE IMPEDANCE SURFACE MODELING

3.1 Application of the Multi-Scale Equivalent Circuit Method

The multifractal planar structure is considered as a discontinuity in the transverse section of the waveguide as presented on figure 3.

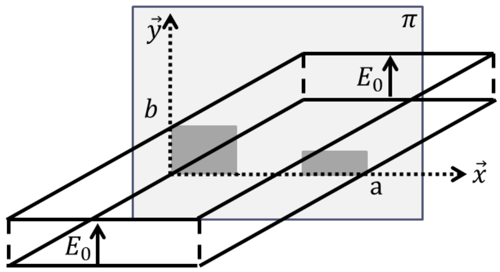

Fig -3: Discontinuity inside rectangular waveguide

The rectangular waveguide with the perfect horizontal electrical conductor (PEC) and perfect vertical magnetic conductor (PMC) is excited by its fundamental mode which is the TEM mode denotedf_0. The electric current density on each side of the plan π inside the waveguide is related to magnetic field by:

$$\vec{J_1} = \vec{H_1} \times \vec{n_1}$$

$$\vec{J_2} = \vec{H_2} \times \vec{n_2}$$

Only the tangential components of the electromagnetic field to be considered. So if S1 and S2 are too close to the discontinuity surface S we can write:

$$\vec{E_1} = \vec{E} = \vec{E_2}$$

$$\vec{H_1} - \vec{H_2} = \vec{J} \times \vec{n_1} \Rightarrow \vec{J} = -(\vec{J_1} + \vec{J_2})$$

As described in [] the equivalent circuit scheme of the electromagnetic problem of figure is schematized by the circuit on the figure 4 below.

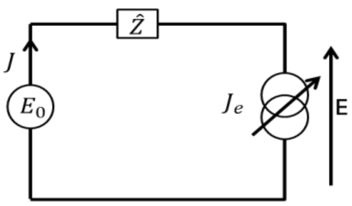

Fig 4: Equivalent electrical circuit model of electromagnetic diffraction problem

Where $\hat{Z} = \sum_{m,n} |f_{m,n}^{TE}\rangle\, z_{m,n}^{TE} \langle f_{mn}^{TE}| + \sum_{mn} |f_{mn}^{TM}\rangle\, z_{mn}^{TM} \langle f_{mn}^{TM}|$ is the impedance operator related to the tangential TE and TM components of the electric field in relation with the impedance mode$z_{m,n}^{TE}$, z_{mn}^{TM} .

$J = -J_e$is the induced current on the metallic parts of the surface discontinuity here is the multifractal planar structure.

By applying the Galerkin method we deduce the surface impedance seen by the fundamental mode of the structure:

$$Z_s(f) = \frac{V_0}{I_0} = \frac{1}{A^t BA}$$

Where $A = \begin{bmatrix} \langle f_0 | g_{e1} \rangle \\ \vdots \\ \langle f_0 | g_{eP} \rangle \end{bmatrix}$,the $g_{ei}\, i \in [1..P]$ are the current trial functions defined on the conductor subdomains of the generator and they verify the limits conditions on the junctions between the waveguide PEC and PMC .

$$B = [B_{pq}]$$

Where$B_{pq} = \sum_{mn} \langle g_{ep} | f_{mn}^{TE+TM} \rangle\, z_{mn}^{TE+TM} \langle f_{mn}^{TE+TM} | g_{eq} \rangle$

$f_{mn}^{TE+TM}$are the (m, n) TE and the TM field components mode. Thez_{mn}^{TE+TM} are the impedance of the (m, n)mode related to the transverse electrical and magnetic fields.

In the aim of enhancing the surface impedance computation of the multifractal structure at infinity the Multiscale technique is used: In the case of multifractal Cantor set, we model on figure 5 the first iteration by twosub surface impedances based on the scaled generator elements like when we apply the IFS. So thesurfaceimpedance of the first iteration is modeled by twosurface impedance of the generator scaled by the IFS contractors:

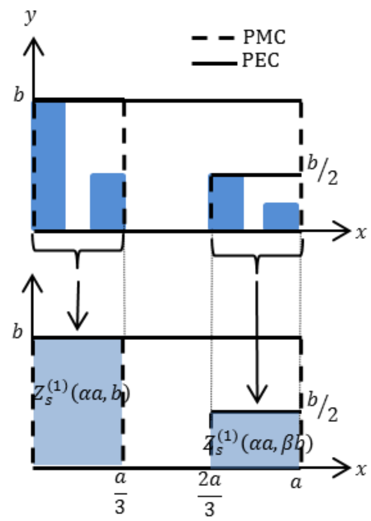

Fig -5: The Multiscale scheme modeling.

The equivalent circuit of electromagnetic problem is shown on figure 6.

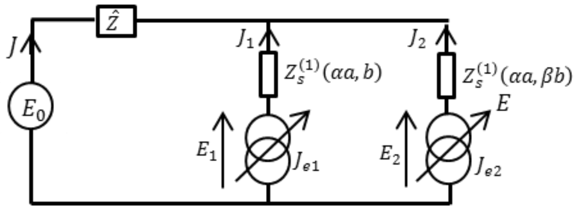

Fig -6: Equivalent electrical circuit model of electromagnetic diffraction problem of Figure 5

To check the validation of the proposed model on figure 6 we compute the surface impedance of the first iteration with two different manners : by applying directly the MS-GEC-MoM to the first iteration: we define trial functions on each metallic parts, the surface impedance is named Zs_direct in the second hand by the model presented at figure 6 where trial function are defined only on the tow domain characterized by their sub surface impedancesof the generator scaled by the IFS, which are previously computed, and it is named Zs_model . On figure 7 we present both of them: Zs_direct and Zs_model. We can deduce that the proposedsurface impedance model is in accordance with thesurface impedance at least for an interval of the frequency simulation

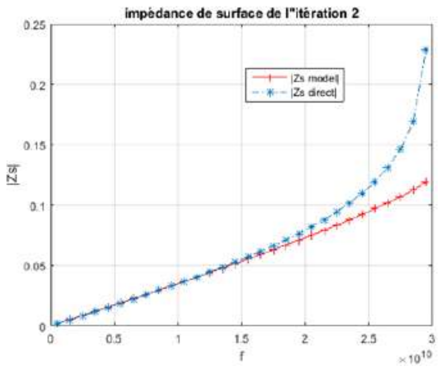

Fig -7: Surface impedance vs model surface impedance

On the frequency interval where there is accordance we can deduce that the surface impedance of the considered iteration1is depending on the previous iteration scaled by the IFS dilatation coefficients: $\alpha_1, \beta_1, \alpha_2, \beta_2$ that means

$$Z_s^{(n-1)}(\alpha_1 a, \beta_1 b) \text{ and } Z_s^{(n-1)}(\alpha_2 a, \beta_2 b)$$

So we can write for a fixed frequency that

$$Z_s^{(n)}(a,b) = f\left(\alpha_1\beta_1 Z_{s1}^{(n-1)}, \alpha_2\beta_2 Z_{s2}^{(n-1)}\right)$$

The function f is a recurrence relation on surface impedance. The gain here is the reduction of matrix dimension by reducing the number of trial functions, in another word, instead of defining current trial functions on each metallic part of the multifractal planar structure they will be defined only on two parts. Second, knowing the function f and the surface impedance of the generator parametrized with $Z_s^{(generator\ )}(\alpha a, \beta b)$ and we can deduce the surface impedance of finite n^{th} iteration by applying the function f recursively. But when fractalization goes to infinity the MS-GEC is inapplicable: first, because the generator will be too small to a point that its surface impedance incomputable; second at infinity of a fractalization level the dimension of the multifractal Cantor set is not an integer as usual in Euclidian geometry so defining current trial function became a hard task and we can say that it is not possible at all, these are the major limitations of the MS-GEC when dealing with fractal and multifractal planar structures.

3.2 Application of the Renormalization Group theory

To overcome the MS-GEC-MoM limitations when dealing with electromagnetic diffraction by multifractal planar structure, in our case the multifractal Cantor set through the surface impedance characterisation we apply the Renormalization Group mechanism resumed in 4 steps below:

The first step is decimation: it consists in regrouping elements of the initial system hierarchically in small groups or blocs at each step of decimation a scale is defined on each bloc we compute the input impedance of the bloc at the considered scale. The second step is the scale changing: instead of studying in details the previous scale (n-1) it will be replaced at the considered scale (n) by an average value of the fluctuation which is here the input impedance so the actual scale will be simplified by reducing details in it.

The first and the second steps of the renormalization group mechanism are shown and explained on figure 4:

In our case the decimation and the scale changing are ensured by the Iterated Function System and the MS-GEC as shown on figure 5.

The third step is the Renormalization relation and effective parameter transformation which consists in replacing real parameters by effective ones in the renormalization relation relating the surface impedance of the considered fractalization level to the previous one in the aim to have a renormalization group transformation independent of scale coefficients.

The relation of renormalization transformation is deduced from the circuit presented on figure 6 by applying the electrical circuit theory the relation is in equation

Inwhich we we replace real parameters by the effective parameters:

$$\left(Z_{s2}^{(n-1)}\right)' = \alpha_2\beta_2 Z_{s2}^{(n-1)}$$

$$\left(Z_{s1}^{(n-1)}\right)' = \alpha_1\beta_1 Z_{s1}^{(n-1)}$$

The fourth and the last step is the fixed point calculation: when applying iteratively to the infinity the renormalization group transformation it will converge to the critical fixed point; as it converges geometrically to a fixed set.

$$Z_s^{'} = Z_{s1}^{'} = \lim_{n\to\infty} \left(Z_{s1}^{(n-1)}\right)^{'}$$

$$Z_s^{'} = Z_{s2}^{'} = \lim_{n\to\infty} \left(Z_{s2}^{(n-1)}\right)^{'}$$

So we can compute the input surface impedance at infinity by solving graphically on figure 8 the equation of the fixed point defined as:

$$Z_s^{'} = f(Z_s^{'}, Z_s^{'})$$

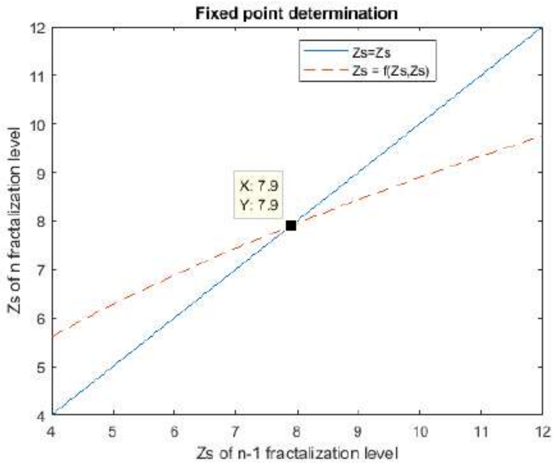

Fig -8: Graphical determination of thefixed point.

We plot on the same graph the curve presented by

$$Z_s^{(n)} = f\left(Z_s^{(n-1)}, Z_s^{(n-1)}\right)$$

And the $Z_s^{(n)} = Z_s^{(n-1)}$

The intersection between the two curves gives fixed point.

Table 1: Surface impedance computation comparison between MS-GEC-MoM and MS-GEC-RGT.

	MS-GEC-MOM modeling	MS-GEC-RGT modeling
Memory	187.4MB	67.3MB
Matrix size	3844	1936
Computing time	0h15m53s	0h8m45s
$\lvert Z_s^9\rvert = \lvert Z_s^\infty\rvert$ (fixed point)	7.82Ω/s	7.9Ω/s

A comparison between the two modeling techniques on table 1 prove that due to the MS-GEC-RGT we reduce the matrix size by 49.63 % leading to save memory usage by 64.08% and computing time by 43.85% with an interesting accuracy in impedance surface value at an infinite fractalization level with an error of 1.02%

3. CONCLUSIONS

Due to their scale invariance fractal and multifractal planar structure are classified as critical phenomenon. So it is suitable to combine the renormalization group theory mechanism to the Multi-Scale Generalized Equivalent Circuit (MS-GEC) to make the simulation of the surface impedance fluctuation of multifractal Cantor set at infinity level of fractalization possible simple and accurate besides saving time, memory and CPU resources.

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