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Electric field on an infinite plane of uniform charge via Laplace and Fourier transforms

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Abstract: *In this paper, we consider two electrostatic systems and compute the magnitude of the electric field at points on the surface for each one. In the first one, the Laplace transform method is used to find the value of the electric field at points on an infinite plane of charge. In the second one, the Integral Fourier transform approach is used to derive the electric field on one plate of a parallel-plate capacitor. It is shown that, the electric field at points on the plate is zero for the first system and is exactly half its value between the plates of the capacitor for the second one. These techniques demonstrate their usefulness in determining the value of a discontinuous function at its point of discontinuity. It also helps undergraduate physics and engineering students to realize the advantage of the application of integral transforms when applied to physical problems.*

Keywords: *Fourier and Laplace transforms, electric field, parallel-plate capacitor.*

1. INTRODUCTION

The problem of determining the electrostatic field in space due to charged surfaces is discussed in introductory textbooks [1, 2]. The usual method for symmetrical charge distribution is through the application of Gauss's law. However, the question of finding the electric field at points on such surfaces (like conducting charged sphere and charged planes) is avoided due to the discontinuity of the electric field at such points. The limits of the electric field as one approaches the surface from both sides are not equal and therefore the electric field is ill-defined on the surface. Students are usually not aware of this ambiguous situation and may assign one of the limiting values for the electric field on the surface. This is not a correct conclusion and one should admit the ambiguity of the electric field on the surface and care must be taken when dealing with such problems. In some situations, one has to assign a value for the electric field at its discontinuity points for the purpose of calculating other physical quantities. For example, Griffiths [3] calculated the electrostatic pressure on a charged conducting surface by using the average value of the electric field between its values inside and just outside the surface, but without rigorous proof, and obtained the well-known result for the pressure [4]. It is important for the students to know this discontinuity of the electric field [5, 6] and learn how to use it and to understand its ambiguity [7, 8]. In the light of this, the present paper deals with two specific systems which are familiar for undergraduate students. In the first system, we consider a uniformly charged large thin plate and derive the electric field at points on the plate using two methods: In the first method, we apply the Laplace transform approach and in the second, we use the expansion of the electric field in terms of Legendre polynomials. In the second system, we consider a parallel-plate capacitor and derive the electric field on one plate through the application of integral Fourier transform.

2. THE ELECTRIC FIELD ON A PLANE OF CHARGE

We consider an infinite plane of positive charge with uniform surface charge density, σ. Assuming the plane to be aligned in the yz-plane, direct application of Gauss law yields the electric field E, at points on both sides of the plane, namely

$$E = \frac{\sigma}{2\varepsilon_0} f(x) = \frac{\sigma}{2\varepsilon_0}\begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \tag{1}$$

where ε_0 is the electric permittivity for free space. Clearly, the electric field is discontinuous at the surface of the plane of charge, since the electric fields on both sides of the plane are in opposite directions. The aim here is to derive the electric field at points on the plane (x=0) through the application of Laplace transform which has been of great use over the years [9-15]. Using a decreasing exponential kernel with the function $f(x)$ in equation (1), consider the following integral;

$$F(s) = \int_{-\infty}^{\infty} e^{-s|x|} f(x)dx \qquad s > 0. \tag{2}$$

Splitting the integrals into two parts and substituting the value of $f(x)$ for each part gives

$$F(s) = \int_{-\infty}^{0} e^{sx}(-1)\,dx + \int_{0}^{\infty} e^{-sx}(1)dx. \tag{3}$$

Letting $x \to -x$ in the first integral, we get

$$F(s) = \int_{\infty}^{0} e^{-sx}dx + \int_{0}^{\infty} e^{-sx}dx. \tag{4}$$

Using the step function,

$$U(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \tag{5}$$

equation (4) could be written as

$$F(s) = -\int_{0}^{\infty} e^{-sx}\,U(x)dx + \int_{0}^{\infty} e^{-sx}U(x)dx. \tag{6}$$

Each integral of the above equation is the Laplace transform of $U(x)$ [16], namely

$$\mathcal{L}\{U(x)\} = \int_0^{\infty} e^{-sx}\, U(x)dx = \frac{1}{s}, \quad (7)$$

so that equation (6) gives

$$F(s) = -\frac{1}{s} + \frac{1}{s} = 0\,. \quad (8)$$

Therefore, with the result of equation (8), equation (2) becomes

$$F(s) = \int_{-\infty}^{\infty} e^{-s|x|} f(x)dx = 0. \quad (9)$$

Note that, in the limit when $s\to\infty$, the kernel in equation (10) goes to zero except at the point of discontinuity (x=0) of the function $f(x)$. In this limiting case, the exponential kernel becomes just a non-zero constant, and therefore the value of the function at its point of discontinuity must be zero in order the integral in equation (9) to vanish. So the conclusion is that $f(0)$=0, which is evidently the average value between $f(x)$ values for x<0 and x>0. Therefore, equation (1) gives $E(0)$=0, which shows that the electric field vanishes at the plane of charge. One concludes that the electric field at discontinuous points is the average value between its limiting values around that point.

Another way to derive our above result is to expand the function $f(x)$ in equation (1) in terms of Legendre polynomials $P_n(x)$ as

$$f(x) = \sum_{n=0}^{\infty} C_n\, P_n(x). \quad (10)$$

Multiplying both sides of equation (10) by $P_m(x)$ and integrating from −1 to +1, we get

$$\int_{-1}^{1} f(x)P_m(x)dx = \frac{2}{2n+1} C_n, \quad (11)$$

where we used the orthonormality

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}.$$

Since the function $f(x)$ is odd, then the integral in equation (11) vanishes for even parity of $P_m(x)$. Therefore, all expansion coefficients C_n vanish for even n, and thus, we have

$$f(x) = \sum_{n=odd}^{\infty} C_n\, P_n(x). \quad (12)$$

Now our result is clear: For odd n, $P_n(0)$=0 so that the right hand side of equation (12) gives zero and therefore, $f(0)$=0. Hence, the electric field vanishes at points on the plane of charge.

3. TWO INFINITE PARALLEL PLANES OF CHARGE

Consider two infinite parallel planes (like parallel-plate capacitor) one carries uniform surface charge density σ and the other $-\sigma$, with separation distance $2a$ between the planes. Direct application of Gauss law gives the electric field $E=\sigma/\varepsilon_0$ between the plates and zero outside, so that the electric field is discontinuous at the plates due to the surface charge density. The object of this section is to find the value of the electric field on each plate through the Fourier integral approach. This approach has been of great applications over the last decade [17-21]. Let the positive plate positioned at $x=-a$ and the negative plate at $x=a$, so that the origin of the coordinate system is at the midway point between the plates. Therefore, the electric field is given by

$$E(x) = \begin{cases} \frac{\sigma}{\varepsilon_0} & |x| < a \\ 0 & |x| > a \end{cases}, \quad (13)$$

which is of rectangular shape of height σ/ε_0 and width $2a$. The electric field is symmetric about the plane x=0, so it suffices to take the cosine integral Fourier transform, $g_c(\omega)$ given by

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a E(x)\cos(\omega x)\, dx = \sqrt{\frac{2}{\pi}}\, \frac{\sigma}{\varepsilon_0} \int_0^a \cos(\omega x)\, dx \quad (14)$$

which immediately gives,

$$g_c(\omega) = \sqrt{\frac{2}{\pi}}\, \frac{\sigma}{\varepsilon_0}\, \frac{\sin(\omega a)}{\omega}. \quad (15)$$

Taking the inverse cosine transform, one gets

$$E(x) = \sqrt{\frac{2}{\pi}} \int_0^a g_c(\omega)\cos(\omega x)\, d\omega = \frac{2}{\pi}\frac{\sigma}{\varepsilon_0} \int_0^a \frac{\sin(\omega a)}{\omega}\cos(\omega x)\, d\omega. \quad (16)$$

Therefore, the integral in equation (16) is

$$\int_0^a \frac{\sin(\omega a)}{\omega}\cos(\omega x)\, d\omega = \frac{\pi\varepsilon_0}{2\sigma}\, E(x). \quad (17)$$

Evaluating the above integral at $x=a$, gives

$$\int_0^a \frac{\sin(\omega a)}{\omega}\cos(\omega a)\, d\omega = \frac{1}{2}\int_0^a \frac{\sin(2\omega a)}{\omega} d\omega = \frac{1}{2}\int_0^a \frac{\sin(u)}{u}\, du = \frac{\pi}{4} \quad (18)$$

where we used, $u=2\omega a$ and

$$\int_0^a \frac{\sin(u)}{u} du = \pi/2.$$

Therefore, equations (17) and (18) yield,

at $x=a$, our final result, namely

$$E(a) = \frac{\sigma}{2\varepsilon_0}. \quad (19)$$

The result in equation (19) shows that the electric field at the plane of charge is the average value between its values inside and outside the plates.

4. CONCLUSION

In this paper, two examples were given to find the electric field on a charged surface, at which the electric field is discontinuous due to the presence of surface charge density. In the first example, the electric field was found on a plane of charge by two methods: The first by using the Laplace transform method and the second by expansion by Legendre polynomials. In the second example, a system of two parallel planes of opposite charges was examined and the electric field on one plane was calculated using the Fourier integral transform. In both examples, it was concluded that the electric field at points on the plane of charge is equal to the average value between its limiting values from both sides of the plane. The two examples presented in this paper have the advantage to help undergraduate students to apply mathematical techniques to physical systems in order to calculate some physical quantities that are not well-understood.

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