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On Hurwitz Polynomials and Positive Functions in Stability Analysis

ZIAD ZAHREDDINE

Mathematics Division, College of Science & Information Systems, Rafik Hariri University, Mechref, Damour, Lebanon

Abstract: *Hurwitz polynomials play an essential role in developing stability criteria for continuous-time systems of differential equations, and studying their properties is a subject of continuous interest. In the present work, we establish two characterizations of Hurwitz polynomials, one in terms of an operation known is stability theory as paraconjugation, and the other in terms of positive functions which are becoming more involved in stability related problems.*

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1. INTRODUCTION

The problem of determining conditions under which all the roots of a given polynomial lie in the left-half plane is one of the fundamental problems in the study of stability of a dynamic system. Such polynomials are called Hurwitz polynomials and they arise in a variety of applications such as control systems, signal processing, circuit analysis, and systems theory. For some references in this respect see [5, 7, and 11].

Most of the techniques used in the derivation of stability criteria involve laborate topics such as: index theory, Sturm chains, Lyaponov equations, or generalized bezoutians. See for example [2, 6, and 8]. The complex analysis approach which we adopt in this work has rarely been used.

In this paper, we put on display some of the characteristics of Hurwitz polynomials using complex analysis techniques. We establish two necessary and sufficient conditions for a polynomial to be Hurwitz, one in terms of a complex operation called paraconjugation, and the other in terms positive functions. It is to be noted that positive functions play a key role in the derivation of stability criteria [1, 4, 9, and 10]. Both Hurwitz polynomials and positive functions have applications in the stability of mechanical and electrical networks [3, and 10].

In section 2, we introduce Hurwitz polynomials and positive functions. In section 3, key lemmas are proved. The main results are established in section 4. In section 5, we end up with some concluding remarks.

2. HURWITZ POLYNOMIALS, PARACONJUGATION AND POSITIVE FUNCTIONS

Definition 1

A non-constant polynomial is a Hurwitz polynomial if all its roots have negative real parts.

Definition 2

The paraconjugate of a rational function $f(s)$ is defined by $f^*(s)=\overline{f(-\bar{s})}$, where $\bar{s}$ denotes the complex conjugate of *s*.

For example, if

$$f(s)=s^n+a_1s^{n-1}+\cdots+a_{n-2}s^2+a_{n-1}s+a_n,$$

then

$$f^*(s)=(-1)^ns^n+(-1)^{n-1}\bar{a}_1s^{n-1}+\cdots+\bar{a}_{n-2}s^2-\bar{a}_{n-1}s+\bar{a}_n.$$

Also, if f is written in the factored form

$$f(s)=(s-s_1)(s-s_2)\cdots(s-s_n)$$

then its paraconjugate can be written as

$$f^*(s)=(-1)^n(s+\bar{s}_1)(s+\bar{s}_2)\cdots(s+\bar{s}_n)$$

Definition 3

A rational function g is said to be positive if $\operatorname{Re} g(s)>0$ whenever $\operatorname{Re} s>0$.

3. BASIC LEMMAS

The following two lemmas are needed to establish the main results.

Lemma 1

Given a non-constant polynomial f and its paraconjugate f^*, then the roots of f^* are mirror reflection of the roots of f with respect to the imaginary axis. In particular, any imaginary root of f or f^* is common to both.

Proof

From the factored forms of f and f^*, it is obvious that s_j is a root of f if and only if $-\bar{s}_j$ is a root of f^*.

But if $s_j=x+iy$ then $-\bar{s}_j=-x+iy$.

Therefore, s_j and $-\bar{s}_j$ are symmetric with respect to the imaginary axis.

Now assume that $s = iy$ is a pure imaginary root of f, then since $-\bar{s} = iy$, it follows that s is also a root of f^*, and that proves the second part of the lemma.

Lemma 2

Suppose

$$\operatorname{Re} s_j < 0, \text{ then } |s - s_j| > |s + \bar{s}_j| \text{ whenever } \operatorname{Re} s > 0.$$

Proof

Assuming

$$\operatorname{Re} s_j < 0, \text{ and } \operatorname{Re} s > 0, \text{then} \operatorname{Re} s_j \cdot \operatorname{Re} s < 0.$$

It therefore follows that

$(s_j + \bar{s}_j)(s + \bar{s}) < 0.$ By expanding, we get

$$-s\bar{s}_j - s_j\bar{s} > ss_j + \bar{s}\,\bar{s}_j.$$

Add to both sides of this inequality the expression $s\,\bar{s} + s_j\bar{s}_j$ to get

$$s\,\bar{s} - s\,\bar{s}_j - s_j\bar{s} + s_j\bar{s}_j > s\,\bar{s} + ss_j + \bar{s}\,\bar{s}_j + s_j\bar{s}_j.$$

The last inequality can be written in the form

$(s - s_j)(\bar{s} - \bar{s}_j) > (s + \bar{s}_j)(\bar{s} + s_j)$, or equivalently

$|s - s_j|^2 > |s + \bar{s}_j|^2$ which implies $|s - s_j| > |s + \bar{s}_j|$.

4. MAIN RESULTS

The following theorem establishes a necessary and sufficient condition for a polynomial to be Hurwitz in terms of the paraconjugate operation.

Theorem1

Assume f is a non-constant polynomial, f and f^* have no common roots. let $g(s) = \dfrac{f^*(s)}{f(s)}$.

Then, f is a Hurwitz polynomial if and only if g maps the right-half plane into the unit circle.

Proof

Suppose f is a Hurwitz polynomial. Write f and its paraconjugate f^* in their factored forms,

Since

$$f(s) = (s - s_1)(s - s_2)\cdots(s - s_n) \text{ and } f^*(s) = (-1)^n(s + \bar{s}_1)(s + \bar{s}_2)\cdots(s + \bar{s}_n)$$

$\operatorname{Re} s_j < 0$, for all $1 \le j \le n$, then by Lemma 2

$$|s - s_j| > |s + \bar{s}_j| \text{ for all } 1 \le j \le n, \text{whenever } \operatorname{Re} s > 0.$$

We therefore obtain

$$|f(s)| > |f^*(s)| \text{ whenever} \operatorname{Re} s > 0.$$

Equivalently, $|g(s)| < 1$.

We proved that, $|g(s)| < 1$ whenever $\operatorname{Re} s > 0$.

In other words, g maps the right-half plane into the unit circle.

Conversely suppose that, $|g(s)| < 1$ whenever $\operatorname{Re} s > 0$, then $|f(s)| > |f^*(s)|$ for $\operatorname{Re} s > 0$.

It follows that f cannot have any roots for Re $s > 0$, and therefore, the only possible roots of f when Re $s \ge 0$ are pure imaginary.

But by Lemma 1, any pure imaginary root of f is also a root of f^*, and that contradicts the assumption that f and f^* have no common roots.

The only option is for f to have only roots with negative real parts, and therefore f is a Hurwitz polynomial.

Theorem 2 establishes a strong correlation between Hurwitz polynomials and positive functions.

Theorem 2

Let f be a non-constant polynomial, f and f^* have no common roots. Define,

$$h(s) = \frac{f(s) - f^*(s)}{f(s) + f^*(s)}.$$

Then f is a Hurwitz polynomial if and only if h is a positive function.

Proof

$$h = \frac{f - f^*}{f + f^*} \Leftrightarrow h = \frac{1 - g}{1 + g} \text{ where } g = \frac{f^*}{f}.$$

$$\text{Obviously } h = \frac{1 - g}{1 + g} \Leftrightarrow g = \frac{1 - h}{1 + h}.$$

By direct computations we can prove that:

$$h + \bar{h} = \frac{2(1 - g\bar{g})}{|g + 1|^2} \text{ and } 1 - g\bar{g} = \frac{2(h + \bar{h})}{|h + 1|^2}.$$

Now suppose that f is a Hurwitz polynomial, then by Theorem 1, this is equivalent to

$$|g(s)| < 1 \text{ whenever } \operatorname{Re} s > 0.$$

But $|g| < 1 \Leftrightarrow g\bar{g} < 1 \Leftrightarrow 1 - g\bar{g} > 0 \Leftrightarrow h + \bar{h} = \dfrac{2(1 - g\bar{g})}{|g + 1|^2} > 0 \Leftrightarrow \operatorname{Re} h > 0.$

Therefore, f is a Hurwitz polynomial if and only if Re $h(s) > 0$ whenever Re $s > 0$.

In other words, f is a Hurwitz polynomial if and only if h is a positive function.

5. CONCLUSION

We established two characterizations of Hurwitz polynomials, one in terms of paraconjugation, and the other in terms of positive functions. Both concepts of Hurwitzness and positiveness play significant roles in the development of stability criteria of continuous-time systems of differential equations, and have direct applications in the stability analysis of electrical and mechanical systems.

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