Universal System for Many-Valued Logic, Based on Splitting Method, and Some of Its Properties

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Abstract: *Some new proof system **UGS** is introduced such that propositional proof system for every version of MVL can be presented in described form. This proof system is based on the splitting method of variables, it is "weak" ones with a "simple strategist" of proof search, but the preference of such systems is the possibility of simplification of proof by choosing the order of splinted variables. We prove also that some of minimal tautologies must derived in **UGS** more harder, than any substitution in it.*

Keywords: *many-valued propositional logic, determinative conjunct, generalization of splitting method, proof complexity, monotonous and strongly monotonous proof systems.*

1. INTRODUCTION

Many-valued logic (MVL), which was created and developed in 1920 first by Łukasiewicz [1], has in the mean time many interesting applications in many fields of Mathematics and Software Engineering, therefore the investigations in area of MVL are very actual.

Two types of universal propositional proof systems were described in [2] such that propositional proof system for every version of MVL can be presented in both of described forms. We introduce in this paper some other universal system ***UGS***, based on splitting method, described in [3] and generalized for two-valued logic in [4]. This type proof system is "weak" ones with a "simple strategist" of proof search and we have investigated the some quantitative properties, related to proof steps in it. In particular, we investigate the relations between the proof steps for minimal tautologies and for results of some substitutions in them. The minimal tautologies, i.e. tautologies, which are not a substitution of a shorter tautology, play main role in proof complexity area. Really all propositional formulaes, investigated in proof complexity theory, are minimal tautologies. There is traditional assumption that minimal tautology must be no harder than any substitution in it. We show that this assumption is wrong for the introduced system ***UGS***: the proof steps of substituted formulas can be less than the proof steps of corresponding minimal tautologies.

This article consists from follow main sections: Introduction, Preliminaries, in which the main notions, materials and methods are given, Main Results, in which the universal system ***UGS*** is described and give the results of the relation between the steps of minimal tautologies and formulas, which are some substitutions in them.

In the end of paper we give Conclusion.

2. PRELIMINARIES

2.1. Main notions of k-valued logic.

Let E_k be the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$. We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from E_k, (may be also propositional constants), parentheses (,), and logical connectives $\&, \vee, \supset, \neg$, every of which can be defined by different mode. Additionaly we use two modes of exponential function p^σ and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^B}$ (for both modes) is formula also.

In the considered logics either only **1** or every of values $\frac{1}{2} \leq \frac{\boldsymbol{i}}{\mathbf{k-1}} \leq 1$ can be fixed as ***designated values***.

Definitions of main logical functions are: (1)$\boldsymbol{p} \vee \boldsymbol{q} = max(p,q)$ or (2)$\boldsymbol{p} \vee \boldsymbol{q} = \min(p+q, 1)$,

(1)$\boldsymbol{p}\&q = min(p,q)$ or (2)$\boldsymbol{p}\&q = \max(\boldsymbol{p} + q - 1, 0)$

For implication we have two following versions:

(1)$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ p \leq q \\ 1-p+q, & for\ p > q \end{cases}$ or

(2) $\mathbf{p} \supset \boldsymbol{q} = \begin{cases} 1, & for \quad p \leq q \\ q, & for \quad p > q \end{cases}$

And for negation two versions also:

(1)$\neg \boldsymbol{p} = 1 - p$ or (2)$\neg \boldsymbol{p} = ((k-1)p + 1)(mod\ k)/(k-1)$

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta} = \frac{i}{k-1} (0 \leq i \leq k\text{-}1)$ we define additionally "exponent" functions:

(1) $\boldsymbol{p^\delta} = (p \supset \delta) \& (\delta \supset p)$ with (1) implication and (2) $\boldsymbol{p^\delta}$ as p with $(k\text{-}1)(1 - \delta)$ (2) negations.

If we fix "**1**" (every of values $\frac{1}{2} \leq \frac{i}{k-1} \leq 1$) as designated value, so a formula φ with variables $p_1, p_2, \dots p_n$ is called ***1-k*-tautology** (***≥1/2-k-tautology***) if for every $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in E_k^n$ assigning δ_j ($1 \leq j \leq n$) to each p_j gives the value 1 (or some value $\frac{i}{k-1} \geq \frac{1}{2}$) of φ.

Sometimes we call ***1-k*-tautology** or ***≥1/2-k-***

tautology simply ***k*-tautology.**

2.2. Determinative Disjunctive Normal Form for MVL

The notions of determinative conjunct is definished for all variants of MVL in **[2].**

For every propositional variable p in k-valued logic $p^0, p^{1/k-1}, \dots, p^{k-2/k-1}$ and p^1 in sense of both exponent modes are the ***literals.*** The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable with different measures of exponents simultaneously).

Replacement-rule are each of the following trivial identities for a propositional formula $\boldsymbol{\psi}$:

for both variants of conjunction and both variants of disjunction

$\varphi\&0 = 0\&\varphi = 0,$ $\varphi \vee 0 = 0 \vee \varphi = \varphi,$ $\varphi\&1 = 1\&\varphi = \varphi,$ $\varphi \vee 1 = 1 \vee \varphi = 1,$

for (1) implication

$\varphi \supset 0 = \bar{\varphi}$ with (1) negation, $0 \supset \varphi = 1,$ $\varphi \supset 1 = 1,$ $1 \supset \varphi = \varphi,$

for (2) implication

$\varphi \supset 1 = 1,$ $0 \supset \varphi = 1,$ $\varphi \supset 0 = \overline{sg}\varphi,$ where $\overline{sg}\varphi$ *is* 0 for φ>0 and 1 for φ=0,

for (1) negation ¬(i/k-1)=1-i/k-1 (0≤i≤k-1), $\neg\boldsymbol{\psi} = \boldsymbol{\psi},$

for (2) negation ¬(i/(k-1))=(i+1)/(k-1) (0≤i≤k-2), $\neg\mathbf{1} = \mathbf{0},$ $\overbrace{\neg\neg\dots\neg}^{k}\boldsymbol{\psi} = \boldsymbol{\psi}.$

Application of a replacement-rule to some word consists in replacing of its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

In [2] the following ***auxiliary relations for replacement*** are introduced as well:

for both variants of conjunction $\left(\varphi\&\frac{i}{k-1}\right) = \left(\frac{i}{k-1}\&\varphi\right) \le \frac{i}{k-1}$ $(1 \le i \le k-2),$

for both variants of disjunction $\left(\varphi\vee\frac{i}{k-1}\right) = \left(\frac{i}{k-1}\vee\varphi\right) \ge \varphi$ $(0 \le i \le k-2),$

for (1) implication $\left(\varphi \supset \frac{i}{k-1}\right) \ge \frac{i}{k-1}$ and $\left(\frac{i}{k-1} \supset \varphi\right) \ge \frac{k-(i+1)}{k-1}$ $(1 \le i \le k-2),$

for (2) implication $\left(\varphi \supset \frac{i}{k-1}\right) \ge \frac{i}{k-1}$ $(1 \le i \le k-$ 2, ik−1⊃φ≥φ 1≤i≤k−1.

Let φ be a propositional formula of k-valued logic, $P = \{p_1, p_2, \dots, p_n\}$ be the set of all variables of φ and $P' = \{p_{i_1}, p_{i_2}, \dots, p_{i_m}\}$ $(1 \le m \le n)$ be some subset of P.

Definition 2.2.1: Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_k^m$, the conjunct $K^{\sigma} = \{p_{i_1}{}^{\sigma_1}, p_{i_2}{}^{\sigma_2}, \dots, p_{i_m}{}^{\sigma_m}\}$ is called $\varphi - \frac{i}{k-1}$-determinative $(0 \le i \le k-1),$ if assigning σ_j $(1 \le j \le m)$ to each p_{i_j} and successively using replacement-rules and, if it is necessary, the auxiliary relations for replacement also, we obtain the value $\frac{i}{k-1}$ of φ independently of the values of the remaining variables.

Every $\varphi - \frac{i}{k-1}$ −determinative conjunct is called also φ-determinative or determinative for φ.

3. MAIN RESULTS

3.1. Definition of universal systems *UGS.*

The splitting method were described at first in [3] for disjunctive normal forms of two-valued classical logic, then was generalized for each formulas of two-valued classical logic in [4]. Now we introduce the generalization of splitting method for MVL.

Let φ be some formula and p be some of its variable. Results of splitting method of formula **φ** by variable **p** ***(splinted variable)*** are the formlas φ[$\boldsymbol{p^{\delta}}$]for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$, which are obtained from φ by assigning $\boldsymbol{\delta}$ to each occurrence of **p** and successively using replacement-rules and, if, it is necessary, the auxiliary relations for replacement also. Note that, in some cases, the formulas φ[$\boldsymbol{p^{\delta}}$]can remain after pointed transformation occurrences of the constant $\boldsymbol{\delta}$ as well.

Generalization of splitting method allow as associate with every formula φ some tree with root, nodes of which are labeled by formulas and edges, labeled by literals. The root is labeled by itself formula φ. If some node is labeled by formula v and α is some its variable, then all of k edges, which going out from this node, are labeled by one of literals $\alpha^{\boldsymbol{\delta}}$ for every $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$, and every of k “sons” of this node is labeled by corresponding formula v[$\alpha^{\boldsymbol{\delta}}$]. Each of the tree’s leafs is labeled with some constant from the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$. The tree, which is constructed for formula φ by described method, we will call ***splitting tree of*** φ in future.

It is obvious, that changing the order of splinted variables in given formula φ, we can obtain the different splitting trees of φ.

For example, if formula φ = $p_1 \supset (p_2 \supset (p_3 \supset (\dots(p_n \supset p_1)\dots)))$, then by choosing as first splinted variable p_1, we obtain the splitting tree with k+1 nodes, only one of them is labeled by φ, and the others are labeled by 1. Really, for p_1=0 or p_1=1 we obtain φ=1, using ***replacement-rules*** for both modes of implication, for the other values of variable p_1 we obtain φ=1, using ***auxiliary relations for replacement ,*** defined for both modes of implication. It is not difficult to see, that if we choose as first splinted variable some p_i (i≠1), then we obtain the splitting tree with at least, than k^2+1 nodes. Note, that every literal ${p_1}^{\delta}$ for each $\boldsymbol{\delta}$ from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$ is φ -1-determinative, and neither from ${p_i}^{\delta}$ for i≠1and $\boldsymbol{\delta} \neq 0$ is φ –determinative.

Remark 3.1. It is easy to see, that if for some k-tautology φ, the minimal number of literals, containing in φ-determinative conjunct, is m, then

each splitting tree of φ has at least $(k^{m+1} - 1)/(k-1)$ nodes,

the length of every its branch is at least k.

The proof system ***UGS*** can be defined as follows: for every formula φ must be constructed some splitting tree and if all tree's leafs are labeled by the value 1 (or by some value $\frac{i}{k-1} \geq \frac{1}{2}$), then formula φ is ***1-k*-tautology** (***≥1/2-k-tautology***), and therefore we can consider every of pointed constants as axioms, and if v is formula, which is label of some splitting tree node, and p is its splinted variable, then the following figure $\frac{v[p^0],\ v[p^{\frac{1}{k-1}}], \ldots, v[p^{\frac{k-2}{k-1}}],\ v[p^1]}{v}$ can be considered as some inference rule, hence every above described splitting tree can be consider as some proof of φ in the system ***UGS***.

Theorem 1. The system ***UGS*** is complete and sound and propositional proof system for every version of MVL can be presented as ***UGS*** system.

Proof is obvious.

3.2. Definitions of proof complexity measure of splitting tree and some quantitative properties of propositional proof systems.

One of the main complexity characteristic, considered in the theory of proof complexity, is ***t-complexity,*** defined as the number of all different formulas in proof.

Let $\boldsymbol{\Phi}$ be a proof system and $\boldsymbol{\varphi}$ be a tautology. By $\mathbf{t}^{\Phi}(\boldsymbol{\varphi})$ is denoted the minimal possible value of $\boldsymbol{t-complexity}$ for all proofs of tautology $\boldsymbol{\varphi}$ in $\boldsymbol{\Phi}$.

The size of a formula $\boldsymbol{\varphi}$ is defined as the number of all logical signs entries in it.

Definition 3.2.1. The tautology is called ***minimal tautology*** of this logic, if it is not a result of some substitution of a shorter tautology of this logic.

For every minimal tautology φ of fixed logic by S(φ) is denoted the set of all tautologies, which are results a substitution in φ.

Definition 3.2.2. The proof system $\boldsymbol{\Phi}$ is called ***t-monotonous***, if for every minimal tautology φ of this system and for every formula ψ from S(φ) $\boldsymbol{t}^{\Phi}(\varphi) \leq \boldsymbol{t}^{\Phi}(\psi)$.

Definition 3.2.3. The proof system $\boldsymbol{\Phi}$ is called ***t-strong monotonous***, if for every non-minimal tautology ψ of this system there is minimal tautology φ of this system such that ψ belong to S(φ) and $\boldsymbol{t}^{\Phi}(\psi) = \boldsymbol{t}^{\Phi}(\varphi)$.

The relation between the ***t-complexities*** of proofs for minimal tautologies and results of a substitutions in them are investigated formerly for some propositional proof system of two-valued classical and non-classical logics. In particular, it is proved in [5], that Frege systems for classical and non-classical logics are no t-monotonous, propositional resolution systems ***RC, RI, RJ*** for classical, intuitionistic and Johansson's logics accordingly are t-strong monotonous, but neither of them is t-monotonous [6].

Theorem 2.

For all variants of MVL the system ***UGS*** is neither t-strong monotonous, not t-monotonous.

Proof is based on the consideration of the following formulas every of which is k-tautology in all versions of MVL.

$A = (p \supset p) \& (s^0 \vee s^{1/k-1} \vee s^{2/k-1} \vee \ldots \vee s^{k-2/k-1} \vee s^1)$ and

$B = (p \supset p) \& ((p \vee p)^0 \vee (p \vee p)^{1/k-1} \vee (p \vee p)^{2/k-1} \vee \ldots \vee (p \vee p)^{k-2/k-1} \vee (p \vee p)^1)$.

It is not difficult to see, that A is minimal tautology, B is not minimal and belong to S(A). $t^{UGS}(B)$ is 2 (formula B itself and constant 1), every A-determinative conjunct contains 2 variables, hence $t^{UGS}(A) > t^{UGS}(B)$ and therefore the system *UGS* is neither t-strong monotonous, not t-monotonous.

4. CONCLUSION

We give the third universal system for all variants of MVL. The preference of such systems is on the one hand the possibility of simplification of proof by choosing the order of splinted variables, and on the other hand automatic receipt of high lower bounds for tautologies with specific properties: minimal numbers of literals in determinative conjunct must be great. The last properties have the universal systems, introduced in [2].

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