Oscillation of a Class of Nonlinear Fractional Differential Equations with Damping Term

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Abstract: *The purpose of this paper is to study the oscillation of the nonlinear fractional differential equations with damping term. Using the techniques of Riccati transformation and inequality, two criterions for the oscillation of the equations are obtained, and the examples are given to verify the relevant results.*

Keywords: damping term; oscillation; fractional differential equation; Riccati transformation

1. INTRODUCTION

In recent years, many scholars have studied the oscillation of various equations including differential equations, difference equations, and dynamic equations on time scales. At present, scholars have made great efforts to study t-he oscillation of these equations and establish some new oscillation criterions for them[1-9],but there are few studies on the oscillation of fractional differential equations[9-11].In this paper, we study the oscillation of

the following nonlinear fractional differential equations.

$$D_t^{\alpha}(r(t)\psi(x(t))F(D_t^{\alpha}x(t))) + p(t)\psi(x(t))G(D_t^{\alpha}x(t)) + q(t)g(x(t)) = Q(t, D_t^{\alpha}x(t), x(t)) \quad (1)$$

$t \in [t_0, \infty), t_0 \geq 0$, $r(t) \in C^{\alpha}([t_0, +\infty), R)$, $p(t), q(t) \in C([t_0, \infty), R)$, F, G, ψ, $g \in C(R, R)$, Q denotes continuous function on $[t_0, \infty) \times R^2$,where $D_t^{\alpha}(.)$ denotes the modified Riemann-Liouville derivative with respect to the variable t, $\alpha \in (0,1)$.

Assume the following conditions are true:

(A_1) $r(t) > 0, p(t) > 0, q(t) > 0, t \in [t_0, +\infty)$;

(A_2) $\forall x(t) \in R$, $0 < \psi(x(t)) \leq k_1$;

(A_3) $\forall y \in R$, $F^2(y) \leq lyF(y)$, $l > 0$;

(A_4) $\forall x \neq 0$, $\frac{g(x(t))}{x(t)} \geq k$;

(A_5) Exicts a function $v(t)$, $\forall x \neq 0$ and $\forall y \in R$, $\frac{Q(t,y,x)}{x} \leq v(t)$

(A_6) Exicts a function $m(t)$, $G(D_t^{\alpha}x(t)) \geq m(t)F(D_t^{\alpha}x(t))$.

2. PRELIMIARIES

Definition 2.1[12]: The definition and important properties of the modified Riemann-Liouville fractional derivative of order α are listed.

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi \quad \alpha \in (0,1) \quad (2)$$

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha} \quad (3)$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha} f(t) + f(t)D_t^{\alpha} g(t) \quad (4)$$

$$D_t^{\alpha} f[g(t)] = f_g'[g(t)]D_t^{\alpha} g(t) = D_t^{\alpha} f[g(t)](g'(t))^{\alpha} \quad (5)$$

Definition2.2:A solution $x(t)$ of (1)is called oscillatory if it has arbitrarily large zeros; other-wise it is called non oscillatory. Equation (1) is called oscillatory only if all of its solutions are oscillatory.

3. MAIN RESULTS

Assume that $D = \{(t,s) : t \geq s \geq t_0\}$, $D_0 = \{(t,s) : t > s \geq t_0\}$, and there exists a function $H \in C(D, R)$,H is called function belong to the class of Ω,if H satisfies following conditions:

(1) $H(t,s) > 0$ on D_0; $\forall t \geq t_0$, $H(t,t) = 0$;

(2) $H(t,s)$ has a continuous non-positive partial derivative for the second variable on D;

(3) $\exists h(t,s) \in C(D, R)$, $\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}$.

For the sake of convenience, we set $\xi_0 = \frac{t_0^\alpha}{\Gamma(1+\alpha)}$, $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $R_+ = (0,+\infty)$.

Theorem 3.1.Assume $(A_1)-(A_6)$ hold, and if there exist $H \in \Omega$ and $\rho \in C^\alpha([t_0,+\infty), R_+)$, such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)[k\tilde{q}(s)\tilde{\rho}(s) - \frac{k_1 l}{4}\tilde{\rho}(s)\tilde{r}(s)F^2 - \tilde{\rho}(s)\tilde{v}(s)]ds = \infty \quad (6)$$

then (1) is oscillatory, where $F(s) = m(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} + \frac{h(t,s)}{\sqrt{H(t,s)}} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}$, $r(t) = \tilde{r}(\xi)$, $q(t) = \tilde{q}(\xi)$, $\rho(t) = \tilde{\rho}(\xi)$, $v(t) = \tilde{v}(\xi)$.

Proof: Assume that (1) has a non oscillatory solution x(t) on $[t_0,+\infty)$.Without loss of generality, we may assume that $x(t) > 0$, for $t \ge t_0$.

Defining $w(t) = \rho(t)\frac{r(t)\psi(x(t))F(D_t^\alpha x(t))}{x(t)}$, $t \ge t_0$,

we get

$$\begin{aligned} D_t^\alpha w(t) &= \frac{r(t)\psi(x(t))F(D_t^\alpha x(t))}{x(t)} D_t^\alpha \rho(t) \\ &\quad + \rho(t) D_t^\alpha \frac{r(t)\psi(x(t))F(D_t^\alpha x(t))}{x(t)} \\ &= \frac{D_t^\alpha \rho(t)}{\rho(t)} w(t) + \rho(t)\frac{Q(t, D_t^\alpha x(t), x(t))}{x(t)} - \\ &\quad \rho(t)p(t)\frac{\psi(x(t))G(D_t^\alpha x(t))}{x(t)} \\ &\quad - \rho(t)\frac{q(t)g(x(t))}{x(t)} \\ &\quad - \rho(t)\frac{r(t)\psi(x(t))F(D_t^\alpha x(t))D_t^\alpha x(t)}{x^2(t)} \\ &\le \frac{D_t^\alpha \rho(t)}{\rho(t)} w(t) + \rho(t)v(t) - m(t)\frac{p(t)}{r(t)} w(t) \\ &\quad - k\rho(t)q(t) - \rho(t)\frac{r(t)\psi(x(t))F^2(D_t^\alpha x(t))}{lx^2(t)} \\ &\le \frac{D_t^\alpha \rho(t)}{\rho(t)} w(t) + \rho(t)v(t) - m(t)\frac{p(t)}{r(t)} w(t) \\ &\quad - k\rho(t)q(t) - \frac{1}{l\rho(t)r(t)k_1} w^2(t) \end{aligned} \quad (7)$$

Letting $w(t) = \tilde{w}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$, and

Using (3) and (5),we obtain $D_t^\alpha \xi(t) = 1$, $D_t^\alpha w(t) = D_t^\alpha \tilde{w}(\xi) = \tilde{w}'(\xi) D_t^\alpha \xi(t) = \tilde{w}'(\xi)$.Similarly, we also have $D_t^\alpha \rho(t) = \tilde{\rho}'(\xi)$, when $\xi \ge \xi_0$.So inequality (7) can be transformed into the following form:

$$\begin{aligned} \tilde{w}'(\xi) &\le \frac{\tilde{\rho}'(\xi)}{\tilde{\rho}(\xi)}\tilde{w}(\xi) + \tilde{\rho}(\xi)\tilde{v}(\xi) - k\tilde{\rho}(\xi)\tilde{q}(\xi) \\ &\quad - \tilde{m}(\xi)\frac{\tilde{p}(\xi)}{\tilde{r}(\xi)}\tilde{w}(\xi) - \frac{1}{k_1 l\tilde{\rho}(\xi)\tilde{r}(\xi)}\tilde{w}^2(\xi) \end{aligned} \quad (8)$$

Substituting ξ with s, and multiplying both sides of the inequality (8) by $H(t,s)$,integrating it with respect to s from t_0 to t,we get that

$$\begin{aligned} \int_{t_0}^{t} H(t,s)\tilde{w}'(s)ds &\le -\int_{t_0}^{t} \frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s)H(t,s)ds \\ &\quad + \int_{t_0}^{t} (\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)} - \tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)})\tilde{w}(s)H(t,s)ds \\ &\quad + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds - \int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds \end{aligned} \quad (9)$$

Moving the last term of (9) to left, we get

$$\begin{aligned} \int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds &\le -\int_{t_0}^{t} H(t,s)\tilde{w}'(s)ds - \int_{t_0}^{t} [\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) \\ &\quad + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds \\ &= H(t,t_0)\tilde{w}(t_0) + \int_{t_0}^{t} \frac{\partial H(t,s)}{\partial s}\tilde{w}(s)ds - \int_{t_0}^{t} [\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) \\ &\quad + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds \\ &= H(t,t_0)\tilde{w}(t_0) - \int_{t_0}^{t} h(t,s)\sqrt{H(t,s)}\tilde{w}(s)ds - \int_{t_0}^{t} [\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) \\ &\quad + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds \end{aligned}$$

Then,

$$\begin{aligned} \int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds &\le H(t,t_0)\tilde{w}(t_0) - \int_{t_0}^{t} [\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) \\ &\quad + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} + \frac{h(t,s)}{\sqrt{H(t,s)}} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds \\ &\quad + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds \end{aligned}$$

$$= H(t,t_0)\tilde{w}(t_0) - \int_{t_0}^{t}[\sqrt{\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}}\tilde{w}(s) + \frac{1}{2}(\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} + \frac{h(t,s)}{\sqrt{H(t,s)}}$$

$$-\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\sqrt{lk_1\tilde{\rho}(s)\tilde{r}(s)}]^2 H(t,s)ds + \int_{t_0}^{t}\frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s)H(t,s)ds$$

$$+\int_{t_0}^{t}H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds$$

$$\le H(t,t_0)\tilde{w}(t_0) + \int_{t_0}^{t}\frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s)H(t,s)ds$$

$$+\int_{t_0}^{t}H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds$$

Then,

$$\int_{t_0}^{t}H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s)$$

$$-\tilde{\rho}(s)\tilde{v}(s)]ds \le H(t,t_0)\tilde{w}(t_0)$$

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s)$$

$$-\tilde{\rho}(s)\tilde{v}(s)]ds \le \tilde{w}(t_0)$$

Therefore,

$$\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s) \qquad (10)$$

$$-\tilde{\rho}(s)\tilde{v}(s)]ds \le \tilde{w}(t_0)$$

Taking the limit of the both sides of as $t \to \infty$,we have

This

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}F^2(s)$$

$$-\tilde{\rho}(s)\tilde{v}(s)]ds < \infty$$

contradicts to (6).If $x(t) < 0$, the proof is similar and hence is omitted. So the proof is complete.

Theorem 3.2.Assume $(A_1)-(A_6)$ hold, and if there exist $H \in \Omega$, $\rho \in C^{\alpha}([t_0,+\infty), R_+)$, $\varphi \in C^{\alpha}([t_0,+\infty), R)$ and $r\varphi \in C^{\alpha}([t_0,+\infty), R)$,such that

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{q}(s)\tilde{\rho}(s) + \tilde{\rho}(s)\frac{\tilde{r}(s)\tilde{\varphi}^2(s)}{k_1 l}$$

$$-\frac{k_1 l}{4}\tilde{\rho}(s)\tilde{r}(s)Q^2 - \tilde{\rho}(s)\tilde{v}(s) - \tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s) \qquad (11)$$

$$-\tilde{\rho}(s)\tilde{\varphi}(s)\tilde{r}'(s)]ds = \infty$$

then (1) is oscillatory, where

$Q(s) = \frac{\tilde{p}(s)}{\tilde{r}(s)}\tilde{m}(s) + \frac{h(t,s)}{\sqrt{H(t,s)}} - 2\frac{\tilde{\varphi}(s)}{k_1 l} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}$, $r(t) = \tilde{r}(\xi)$, $q(t) = \tilde{q}(\xi)$, $\rho(t) = \tilde{\rho}(\xi)$, $v(t) = \tilde{v}(\xi)$, $\varphi(t) = \tilde{\varphi}(\xi)$.

Proof: Suppose that (1) has a non oscillatory solution x(t) on $[t_0,+\infty)$ .Without loss of generality, we may assume that $x(t) > 0$,for $t \ge t_0$ .The Riccati transform is defined below

$$w(t) = \rho(t)[\frac{r(t)\psi(x(t))F(D_t^{\alpha}x(t))}{x(t)} + r(t)\varphi(t)]$$

Then we get

$$D_t^{\alpha}w(t) = \frac{r(t)\psi(x(t))F(D_t^{\alpha}x(t))}{x(t)}D_t^{\alpha}\rho(t)$$

$$+\rho(t)D_t^{\alpha}\frac{r(t)\psi(x(t))F(D_t^{\alpha}x(t))}{x(t)} +$$

$$\rho(t)D_t^{\alpha}(r(t)\varphi(t)) + r(t)\varphi(t)D_t^{\alpha}\rho(t)$$

$$= D_t^{\alpha}\rho(t)(\frac{w(t)}{\rho(t)} - r(t)\varphi(t)) + \rho(t)\frac{Q(t,D_t^{\alpha}x(t),x(t))}{x(t)}$$

$$-\rho(t)p(t)\frac{\psi(x(t))G(D_t^{\alpha}x(t))}{x(t)} - \rho(t)\frac{q(t)g(x(t))}{x(t)}$$

$$-\rho(t)\frac{r(t)\psi(x(t))F(D_t^{\alpha}x(t))D_t^{\alpha}x(t)}{x^2(t)}$$

$$+\rho(t)r(t)D_t^{\alpha}\varphi(t) + \rho(t)\varphi(t)D_t^{\alpha}r(t)$$

$$+r(t)\varphi(t)D_t^{\alpha}\rho(t)$$

$$\le \frac{D_t^{\alpha}\rho(t)}{\rho(t)}w(t) + \rho(t)v(t) - m(t)\frac{p(t)}{r(t)}w(t) - k\rho(t)q(t)$$

$$-\rho(t)\frac{r(t)\psi(x(t))F^2(D_t^{\alpha}x(t))}{lx^2(t)} + \rho(t)r(t)D_t^{\alpha}\varphi(t)$$

$$+\rho(t)\varphi(t)D_t^{\alpha}r(t)$$

$$\le \frac{D_t^{\alpha}\rho(t)}{\rho(t)}w(t) + \rho(t)v(t) - m(t)\frac{p(t)}{r(t)}w(t)$$

$$-k\rho(t)q(t) - \rho(t)\frac{(\frac{w(t)}{\rho(t)} - r(t)\varphi(t))^2}{lr(t)k_1}$$

$$+\rho(t)r(t)D_t^{\alpha}\varphi(t) + \rho(t)\varphi(t)D_t^{\alpha}r(t)$$

$$= \frac{D_t^{\alpha}\rho(t)}{\rho(t)}w(t) + \rho(t)v(t) - m(t)\frac{p(t)}{r(t)}w(t) - k\rho(t)q(t)$$

$$-\frac{1}{l\rho(t)r(t)k_1}w^2(t) + 2\frac{\varphi(t)}{lk_1}w(t) - \rho(t)\frac{r(t)\varphi^2(t)}{lk_1} \qquad (12)$$

$$+\rho(t)r(t)D_t^{\alpha}\varphi(t) + \rho(t)\varphi(t)D_t^{\alpha}r(t)$$

Similarly, the inequality (12) can be transformed into the following form

$$\tilde{w}'(\xi) \le \tilde{\rho}(\xi)\tilde{v}(\xi) - k\tilde{\rho}(\xi)\tilde{q}(\xi) - [\frac{1}{k_1 l\tilde{\rho}(\xi)\tilde{r}(\xi)}\tilde{w}^2(\xi) + \qquad (13)$$

$$(\tilde{m}(\xi)\frac{\tilde{p}(\xi)}{\tilde{r}(\xi)} - 2\frac{\tilde{\varphi}(\xi)}{k_1 l} - \frac{\tilde{\rho}'(\xi)}{\tilde{\rho}(\xi)})\tilde{w}(\xi)] - \tilde{\rho}(\zeta)\frac{\tilde{\varphi}^2(\xi)\tilde{r}(\xi)}{k_1 l}$$

$$+\tilde{\rho}(\xi)\tilde{r}(\xi)\tilde{\varphi}'(\xi) + \tilde{\rho}(\xi)\tilde{\varphi}(\xi)\tilde{r}'(\xi)$$

Next substituting ξ with s, and multiplying both sides of

the inequality (13) by $H(t,s)$,then integrating it with respect to s from t_0 to t ,we get that

$$\int_{t_0}^{t} H(t,s)\tilde{w}'(s)ds \le -\int_{t_0}^{t} \frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s)H(t,s)ds + \int_{t_0}^{t}(\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)} + 2\frac{\tilde{\varphi}(s)}{k_1 l} - \tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)})$$
$$\tilde{w}(s)H(t,s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds - \int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds -$$
$$\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l}ds +$$
$$\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{\varphi}(s)\tilde{r}'(s)ds$$

So,

$$\int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds \le -\int_{t_0}^{t} H(t,s)\tilde{w}'(s)ds - \int_{t_0}^{t}[\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)}$$
$$-2\frac{\tilde{\varphi}(s)}{k_1 l} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds - \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\tilde{r}(s)\tilde{\varphi}^2(s)}{k_1 l}ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{\varphi}(s)\tilde{r}'(s)ds$$

$$= H(t,t_0)\tilde{w}(t_0) + \int_{t_0}^{t}\frac{\partial H(t,s)}{\partial s}\tilde{w}(s)ds - \int_{t_0}^{t}[\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)}$$
$$-2\frac{\tilde{\varphi}(s)}{k_1 l} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds - \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\tilde{r}(s)\tilde{\varphi}^2(s)}{k_1 l}ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds +$$
$$\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)ds$$

Thus,

$$\int_{t_0}^{t} H(t,s)k\tilde{\rho}(s)\tilde{q}(s)ds \le H(t,t_0)w(t_0) - \int_{t_0}^{t}[\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}\tilde{w}^2(s) + (\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} +$$
$$\frac{h(t,s)}{\sqrt{H(t,s)}} - 2\frac{\tilde{\varphi}(s)}{k_1 l} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\tilde{w}(s)]H(t,s)ds - \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\varphi^2(s)\tilde{r}(s)}{k_1 l}ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)ds$$

$$= H(t,t_0)\tilde{w}(t_0) - \int_{t_0}^{t}[\sqrt{\frac{1}{k_1 l\tilde{\rho}(s)\tilde{r}(s)}}\tilde{w}(s) + \frac{1}{2}(\tilde{m}(s)\frac{\tilde{p}(s)}{\tilde{r}(s)} + \frac{h(t,s)}{\sqrt{H(t,s)}}$$
$$-2\frac{\tilde{\varphi}(s)}{k_1 l} - \frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)})\sqrt{lk_1\tilde{\rho}(s)\tilde{r}(s)}]^2 H(t,s)ds + \int_{t_0}^{t}\frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s)$$
$$\times H(t,s)ds - \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l}ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds$$
$$+\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)ds$$

$$\le H(t,t_0)\tilde{w}(t_0) + \int_{t_0}^{t}\frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s)H(t,s)ds +$$
$$\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{v}(s)ds - \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l}ds +$$
$$\int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)ds + \int_{t_0}^{t} H(t,s)\tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)ds$$

Then,

$$\int_{t_0}^{t} H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) + \tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l} - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s) -$$
$$\tilde{\rho}(s)\tilde{v}(s) - \tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s) - \tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)]ds \le H(t,t_0)\tilde{w}(t_0)$$

Diving the both side of the above inequality by $H(t,t_0)$,we have

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) + \tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l} - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s)$$
$$-\tilde{\rho}(s)\tilde{v}(s) - \tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s) - \tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)]ds \le \tilde{w}(t_0)$$

Furthermore,

$$\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) + \tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l} - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s)$$
$$-\tilde{\rho}(s)\tilde{v}(s) - \tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s) - \tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)]ds \le \tilde{w}(t_0)$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)[k\tilde{\rho}(s)\tilde{q}(s) + \tilde{\rho}(s)\frac{\tilde{\varphi}^2(s)\tilde{r}(s)}{k_1 l} - \frac{k_1 l\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s) \quad (14)$$
$$-\tilde{\rho}(s)\tilde{v}(s) - \tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s) - \tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)]ds < \infty$$

which contradicts to (11).So the proof is complete.

4. EXAMPLE

1. Consider the equation below

$$D_t^{\frac{1}{2}}(t^2(1+e^{-|x|})\frac{|D_t^{\frac{1}{2}}x(t)|}{1+(D_t^{\frac{1}{2}}x(t))^2})+t(1+e^{-|x|})|D_t^{\frac{1}{2}}x(t)|+(1+3t^2)\frac{x(t)(2+x^2(t))}{1+x^2(t)} = \sin t\cos(D_t^{\frac{1}{2}}x(t))x(t) \quad (15)$$

where $t\geq 1$, $r(t)=t^2$, $p(t)=t$, $q(t)=1+3t^2$, $m(t)=1$,

$t_0=1$, $k_1=2$, $k=1$, $l=1$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{q}(s)\tilde{\rho}(s)-\frac{k_1 l}{4}\tilde{\rho}(s)\tilde{r}(s)F^2 -\tilde{\rho}(s)\tilde{v}(s)]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[(1+3s^2)-\frac{2\times 1}{4}s^2\times (\frac{2}{\sqrt{(t-s)^2}}+\frac{1}{s})^2-1]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[(1+3s^2)-\frac{1}{2}s^2\times (\frac{4}{(t-s)^2}+\frac{4}{s\sqrt{(t-s)^2}}+\frac{1}{s^2})-1]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[3s^2-s^2\frac{2}{(t-s)^2}-\frac{2s}{\sqrt{(t-s)^2}}-\frac{1}{2}]ds=\infty$$

From theorem 3.1,the equation (15） is oscillatory.

2.Consider the equation below

$$D_t^{\frac{1}{2}}(t^2(1+e^{-|x|})\frac{|D_t^{\frac{1}{2}}x(t)|}{1+(D_t^{\frac{1}{2}}x(t))^2})+2t^3(1+e^{-|x|})|D_t^{\frac{1}{2}}x(t)|+(1+3t^2+t^5)\frac{x(t)(2+x^2(t))}{1+x^2(t)}=\sin t\cos(D_t^{\frac{1}{2}}x(t))x(t) \quad (16)$$

$t\geq 1$, $r(t)=t^2$, $p(t)=2t^3$, $q(t)=1+3t^2+t^5$, $m(t)=1$, $k_1=2$,

$k=1$, $l=1$, $t_0=1$, $\varphi(t)=t$, $\rho(t)=1$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)[k\tilde{\rho}(s)\tilde{q}(s)+\tilde{\rho}(s)\frac{\varphi^2(s)\tilde{r}(s)}{k_1 l}-\frac{k_1\tilde{\rho}(s)\tilde{r}(s)}{4}Q^2(s) -\tilde{\rho}(s)\tilde{v}(s)-\tilde{\rho}(s)\tilde{r}(s)\tilde{\varphi}'(s)-\tilde{\rho}(s)\tilde{r}'(s)\tilde{\varphi}(s)]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[1+3s^2+s^5+\frac{s^4}{2}-\frac{2\times 1}{4}s^2(\frac{2}{\sqrt{(t-s)^2}}+s)^2-1-3s^2]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[s^5+\frac{s^4}{2}-\frac{1}{2}s^2(\frac{4}{(t-s)^2}+\frac{4s}{\sqrt{(t-s)^2}}+s^2)]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t(t-s)^2[s^5+\frac{s^4}{2}-s^2\frac{2}{(t-s)^2} -\frac{2s^3}{\sqrt{(t-s)^2}}-\frac{1}{2}s^4]ds$$

$$=\infty$$

From theorem 3.2, the equation (16)is oscillatory.

5. CONCLUSIONS

In this paper, two criterions for determining the oscillation of a fractional differential equation are given. The validity is verified by two examples.

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