International Journal of Innovative Studies in Sciences and Engineering Technology (IJISSET)

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 4 Issue: 9 | 2018*

The Nonpolynomial Spline Solution of Singularly Perturbed Second Order Two Point Boundary Value Problems

Fathi.A. Abd El-Salam [1], Zaki Ahmed Zaki[2,3]

[1]*Mathematics Engineering and Physics Department, Faculty of Engineering in Shoubra, BenhaUniversity, Cairo, Egypt*
[2]*Mathematics Engineering and Physics Department, Faculty of Engineering in Shoubra, BenhaUniversity, Cairo, Egypt*
[3]*Electrical Engineering Department, Faculty of Engineering, Jouf University, Al-jouf , Saudi Arabia*

Abstract:-*In this paper, we develop a six order accurate method based on quartic nonpolynomial spline function at midknots for the numerical solution of singularly perturbed second order two point boundary value problems. Using this spline function a few consistency relations are derived for approximating the solution of the problem. The proposed approach gives better approximation than other existing spline methods and finite difference methods .Convergence analysis of the proposed method is discussed. Two numerical examples are included to illustrate the practical usefulness of our method.*

Keywords:*Quartic nonpolynomial spline, two point boundary value problems, singularly perturbed*

1. INTRODUCTION

We consider the following singularly perturbed second order two point boundary value problem of the form:

$-\epsilon y^{(2)} + f(x)\, y = g(x)\ \ , f(x) > 0\ ,\ \ x \,\epsilon\, [a,b]$(1)

Subject to the boundary conditions

$y(a) = \alpha_0\ , y(b) = \alpha_1$ (2)

Where $f(x)$and $g(x)$are smooth real valued functions in $[a,b]$, α_0 , α_1 are finite real constants and ϵ is a small positive parameter such that $0 < \epsilon \ll 1$. The numerical treatment of singular perturbation problems has received a great deal of attention in the recent past. The problem in which a small parameter multiplies the highest order derivative arises in various fields of science and engineering, for instance fluid mechanics, fluid dynamics, optimal control, chemical reactor theory, etc. A variety of numerical methods are available in the literature to solve that problem for details one may refer to survey article by Kadalbajoo and Patidar[1],various authors have used finite difference methods, finite element methods for the solution of such problems numerically. Hegarty et al. [2] and Niijima [3] established uniformly convergent second order finite difference schemes. Bogalaev[4]used finite element framework and achieved uniform first order accuracy at the nodes.Surla and Stojanovic [5] generated a difference scheme via spline in tension and achieved the error estimate. Mohanty and Navnitjha [6] have developed compression spline based numerical methods for solution of the problem. Aziz and Khan [7] solved this problem by quintic polynomial spline method, Khan et al. [8] used fifth order uniform mesh difference scheme using sexticspline. Rashidinia et al.[9] have solved this problem using spline in compression. Kadalbajoo and Aggarawal [10] considered B-spline collocation to generate second order convergent method for solving this problem. Ramadan et al. [11] have solved this problem using nonpolynomialquintic spline. The aim of this paper is to construct a new spline method based on a nonpolynomial spline function that has a polynomial part and a trigonometric part to develop numerical method for obtaining smooth approximations for the solution of the system (1) and (2)

1.1 Derivation of the Method

We introduce a finite set of grid points xi by dividing the interval $[a,b]$ into n equal parts.

$$x_i = a + ih\ ,\qquad i = 0, 1, \dots, n$$

$x_0 = a\ \ , x_n = b$ *and* $h = \frac{b-a}{n}$(3)

Let $y(x)$ be the exact solution of the system (1) and (2) and Si be an approximation to $y_i = y(x_i)$obtained by the spline function $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) .Each nonpolynomial spline segment $Q_i(x)$has the form

$$Q_i(x) = a_i \sin k\,(x - x_i) + b_i \cos k\,(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i) + e_i,$$

$i = 0, 1, 2, \dots, n-1$(4)

Wherea_i ,b_i,c_i,d_i and e_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method . Eq. (4) reduces to quartic polynomial spline function in$[a,b]$ when $k \to 0$ choosing the spline function in this form will enable us to generalize other existing methods by arbitrary choices of the parameters α, β, and γ which will be defined later in the end of this section.

Thus, our quartic nonpolynomial spline is now defined by the relations: $(i)\ S(x) = Q_i(x)\ ,\ x\,\epsilon\,[x_i, x_{i+1}], i = 0, 1, \dots, n-1$

$(ii)\ S(x)\ \epsilon\ C^{\infty}[a,b]$(5)

First, we develop expressions for the five coefficients of (4) in terms of $S_{i+1/2}, D_i, M_{i+1/2}, T_i$ *and* $W_{i+1/2}$, *where*

$$(i) Q_i\left(x_{i+1/2}\right) = S_{i+1/2}, Q_i^{(1)}(x_i) = D_i$$

$$(ii) Q_i^{(2)}\left(x_{i+1/2}\right) = M_{i+1/2} \qquad (6)$$

$$(iii) Q_i^{(3)}(x_i) = T_i \;\; ; \; Q_i^{(4)}\left(x_{i+1/2}\right) = W_{i+1/2}$$

We obtain via straight forward calculations the following expressions:

$$a_i = \frac{-T_i}{k^3}$$

$$b_i = \frac{T_i}{k^3}\tan(\theta/2) + \frac{1}{k^4\cos(\theta/2)}W_{i+1/2}$$

$$c_i = \frac{1}{2}M_{i+1/2} + \frac{1}{2k^2}W_{i+1/2} \;\; ; \;\; d_i = D_i + \frac{T_i}{k^3} \;\; e_i = S_{i+1/2} - \frac{h}{2}D_i - \frac{h^2}{8}M_{i+1/2} - \frac{h}{2k^2}T_i - \left[\frac{1}{k^4} + \frac{h^2}{8k^2}\right]W_{i+1/2} \quad (7)$$

Where $\theta = kh$ *and* $i = 0, 1, 2, \ldots, n-1$

Now using the continuity (ii) in (5) that is the continuity of quartic nonpolynomial spline $S(x)$ *and its derivatives up to order three are involved at the point* (x_i, S_i) *where the two quartics* $Q_{i-1}(x)$ *and* $Q_i(x)$ *join. Thus* $Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i)$, $m = 0,1,2$ *and* 3.

Using Eqs. (6) And (7) yield the following consistency relations:

$$\frac{h}{2}[D_i + D_{i-1}] = \left(S_{i+1/2} - S_{i-1/2}\right) - \frac{h^2}{8}\left[M_{i+1/2} + 3M_{i-1/2}\right] + \left[\frac{\tan\theta/2}{k^3} - \frac{h}{2k^2}\right][T_i + T_{i-1}]$$

$$[T_i + T_{i-1}] + \left[\frac{1}{k^4\cos(\theta/2)} - \frac{1}{k^4} - \frac{h^2}{8k^2}\right]W_{i+1/2} + \left[\frac{1}{k^4} - \frac{\cos(\theta)}{k^4\cos(\theta/2)} - \frac{3h^2}{8k^2}\right]W_{i-1/2} \quad (8)$$

$$\frac{h}{2}[D_i - D_{i-1}] = \frac{h^2}{2}M_{i-1/2} + \left[\frac{h^2}{2k^2} - \frac{h\sin(\theta/2)}{k^3}\right]W_{i-1/2} \quad (9)$$

$$\frac{\tan(\theta/2)}{k}[T_i + T_{i-1}] = \left[M_{i+1/2} - M_{i-1/2}\right] + \left[\frac{-1+\cos(\theta/2)}{k^2\cos(\theta/2)}\right]W_{i+1/2} + \left[\frac{\cos\theta - \cos(\theta/2)}{k^2\cos(\theta/2)}\right]W_{i-1/2} \quad (10)$$

$$\frac{\tan(\theta/2)}{k}[T_i - T_{i-1}] = \frac{2\sin^2(\theta/2)}{k^2\cos(\theta/2)}W_{i-1/2} \quad (11)$$

Adding Eqs. (8) And (9) then use equation (10), it follows that

$$hD_i = \left(S_{i+1/2} - S_{i-1/2}\right) + \left[\frac{1}{k^2} - \frac{h}{2k\tan(\theta/2)} - \frac{h^2}{8}\right]\left(M_{i+1/2} - M_{i-1/2}\right) + \left[\frac{h^2}{8k^2} + \frac{h}{2k^3\tan(\theta/2)} - \frac{h}{2k^3\sin(\theta/2)}\right]\left(W_{i-1/2} - W_{i+1/2}\right) \quad (12)$$

Adding Eqs. (10) and (11), it follows that

$$T_i = \frac{k}{2\tan(\theta/2)}\left(M_{i+1/2} - M_{i-1/2}\right) + \left(\frac{1}{2k\sin\theta/2} - \right.$$

12k tanθ2Wi−12−Wi+12

(13)

By eliminating D's from Eqs (9) and (12), we get

$$\left(\frac{6h^2}{8k^2} - \frac{h\sin\theta/2}{k^3} + \frac{h[\cos^2\theta/2 - \cos\theta/2]}{k^3\sin\theta/2}\right)W_{i-1/2} = \left(\frac{-h^2}{8k^2} + \frac{-h(\cos\theta/2 - 1)}{2k^3\sin\theta/2}\right)\left(W_{i-3/2} + W_{i+1/2}\right) + \left(\frac{-h^2}{8} + \frac{1}{k^2} - \frac{h}{2k\tan\theta/2}\right)\left(M_{i-3/2} + M_{i+1/2}\right) + \left(\frac{-6h^2}{8} - \frac{2}{k^2} + \frac{h}{k\tan\theta/2}\right)M_{i-1/2} + \left(S_{i-3/2} - 2S_{i-1/2} + S_{i+1/2}\right) \quad (14)$$

Also, by eliminating T's from Eqs (11) and (13), we get

$$W_{i-3/2} + W_{i+1/2} = \left(\frac{k^2\cos\theta/2}{1-\cos\theta/2}\right)\left(M_{i-3/2} - 2M_{i-1/2} + M_{i+1/2}\right) - \left(\frac{2\cos\theta/2 - 2\cos\theta}{1-\cos\theta/2}\right)W_{i-1/2} - \left(\frac{2\cos\theta/2 - 2\cos\theta}{1-\cos\theta/2}\right)W_{i-1/2} \quad (15)$$

Substituting the value of $(W_{i-3/2} + W_{i+1/2})$ *from Eq. (15) into Eq. (14), then from the resulting equation we set two other expressions for* $W_{i-3/2}$ *and* $W_{i+1/2}$ *(increasing and decreasing the indices by one)then substitute these expressions in Eq. (15) ,we get*

$$\omega\left(S_{i-5/2} + S_{i+3/2}\right) + \rho\left(S_{i-3/2} + S_{i+1/2}\right) + \sigma S_{i-1/2} = h^2\left[\alpha\left(M_{i-5/2} + M_{i+3/2}\right) + \beta\left(M_{i-3/2} + M_{i+1/2}\right) + \gamma M_{i-1/2}\right], \; i = 3, 4, \ldots\ldots, n-2 \quad (16)$$

Where $-\epsilon M_i = -f_i S_i + g_i$ *,with* $f_i = f(x_i)$, $g_i = g(x_i)$ *and the six parameters* ω, ρ, σ, α, β *and* γ *are given by*

$\omega = \frac{1-\cos\theta/2}{\theta^2\cos\theta/2}$, $\rho = \frac{4\cos\theta/2-2\cos\theta\ -2}{\theta^2\cos\theta/2}$

, $\sigma = \frac{2-6\cos\theta/2+4\cos\theta}{\theta^2\cos\theta/2}$

$$\alpha = \frac{-8\ +\theta^2+8\cos\theta/2}{8\,\theta^4\cos\theta/2}$$

$$\beta = \frac{-24+16\cos\theta/2-3\theta^2-8\cos\theta+\theta^2\cos\theta}{4\,\theta^4\,(1-\cos\theta/2)} + \frac{(8-\theta^2)\cos\theta+8+3\theta^2}{4\,\theta^4(1-\cos\theta/2)\cos\theta/2}$$

$$\gamma = \frac{32-\theta^2+(16+6\,\theta^2)\cos\theta-24\cos\theta/2}{4\,\theta^4\,(1-\cos\theta/2)} + \frac{(\theta^2-8)-\cos\theta\,(16+6\,\theta^2)}{4\,\theta^4(1-\cos\theta/2)\cos\theta/2}$$

Remark I

For $(\omega,\ \rho,\ \sigma,\ \alpha,\beta,\gamma) = \left(1,4,-10,\frac{1}{48},\frac{76}{48},\frac{230}{48}\right)$

Then our scheme (16) reduced to the scheme in in [12, 13].

The relation (16) gives (n-4) linear algebraic equations in the (n) unknowns $S_{i+1/2}, i = 0,1,2,\dots\dots,n-1$, *so we need four more equations, two at each end of the range of integration for direct computation of* $S_{i+1/2}$. *These four equations are deduced by Taylor series along with the method of undetermined coefficients.*

$$\sigma_0 S_{1/2}+\sigma_1 S_{3/2}+\sigma_2 S_{5/2}+\sigma_3 S_{7/2} = -a_0 S_0 + h^2\left[w_0 M_0+\sum_{i=1}^{i=6} w_i M_{i-\left(\frac{1}{2}\right)}\right], for\ i = 1 \quad (17)$$

$$\sigma_4 S_{1/2}+\sigma_5 S_{3/2}+\sigma_6 S_{5/2}+\sigma_7 S_{7/2}+\sigma_8 S_{9/2} = -b_0 S_0 + h^2\left[\textstyle\sum_{i=1}^{i=7} w_{i+6} M_{i-(1/2)}\right], for\ i = 2 \quad (18)$$

$$\sigma_8 S_{n-\frac{7}{2}}+\sigma_7 S_{n-\frac{7}{2}}+\sigma_6 S_{n-\frac{5}{2}}+\sigma_5 S_{n-\frac{3}{2}} +\sigma_4 S_{n-\frac{1}{2}} = -b_0 S_n + h^2\left[\sum_{i=1}^{i=7} w_{i+6} M_{n-i+\left(\frac{1}{2}\right)}\right], for\ i = n-1 \quad (19)$$

$$\sigma_3 S_{n-7/2}+\sigma_2 S_{n-5/2}+\sigma_1 S_{n-3/2}+\sigma_0 S_{n-1/2} = -a_0 S_n + h^2\left[w_0 M_n+\textstyle\sum_{i=1}^{i=6} w_i M_{n-i+(1/2)}\right], for\ i = n \quad (20)$$

Where a_0, b_0, $\sigma_i^{'}s$, w_i *'s will be determined later to get the required order of accuracy. The local truncation errors* $t_i, i = 1,2,\dots\dots,n$ *associated with the scheme (16-20) can be obtained as follows: first we rewrite the scheme (16-20) in the form:*

$$\omega\left(y_{i-5/2}+y_{i+3/2}\right)+\rho\left(y_{i-3/2}+y_{i+1/2}\right)+\sigma y_{i-1/2} = h^2\left[\alpha\left(y^{(2)}{}_{i-5/2}+\ y^{(2)}{}_{i+3/2}\right)+\beta\left(y^{(2)}{}_{i-3/2}+ y^{(2)}{}_{i+1/2}\right)+\gamma y^{(2)}{}_{i-1/2}\right]+t_i,\quad i = 3,..,n-2 \quad (21)$$

$$\sigma_0 y_{\frac{1}{2}}+\sigma_1 y_{\frac{3}{2}}+\sigma_2 y_{\frac{5}{2}}+\sigma_3 y_{\frac{7}{2}} = -a_0 y_0 + h^2\left[w_0 y^{(2)}{}_0+\textstyle\sum_{i=1}^{i=6} w_i y^{(2)}{}_{i-(1/2)}\right]+t_i, for\ i = 1 \quad (22)$$

$$\sigma_4 y_{\frac{1}{2}}+\sigma_5 y_{\frac{3}{2}}+\sigma_6 y_{\frac{5}{2}}+\sigma_7 y_{\frac{7}{2}}+\sigma_8 y_{\frac{9}{2}} = -b_0 y_0 + h^2\left[\textstyle\sum_{i=1}^{i=7} w_{i+6} y^{(2)}{}_{i-(1/2)}\right]+t_i, for\ i = 2 \quad (23)$$

$$\sigma_8 y_{n-\frac{9}{2}}+\sigma_7 y_{n-\frac{7}{2}}+\sigma_6 y_{n-\frac{5}{2}}+ \sigma_5 y_{n-\frac{3}{2}}+\sigma_4 y_{n-\frac{1}{2}} = -b_0 y_n + h^2\left[\textstyle\sum_{i=1}^{i=7} w_{i+6} y^{(2)}{}_{n-i+\left(\frac{1}{2}\right)}\right]+t_i,\quad for\ i = n-1 \quad (24)$$

$$\sigma_3 y_{n-7/2}+\sigma_2 y_{n-5/2}+\sigma_1 y_{n-3/2}+\sigma_0 y_{n-1/2} = -a_0 y_n + h^2\left[w_0 y^{(2)}{}_n+\textstyle\sum_{i=1}^{i=6} w_i y^{(2)}{}_{n-i+(1/2)}\right]+ t_i,\ \ for\ i = n \quad (25)$$

The terms $y_{i-1/2}$ *and* $y^{(2)}_{i-1/2}, \dots\dots\dots$ *in Eq. (21) are expanded around the point* x_i *using Taylor series and the expressions for* $t_i, i = 3,4,\dots,n-2$ *can be obtained. Also, expressions for* $t_i, i = 1,2,n-1,n$ *are obtained in a similar manner by expanding around* x_0, *for* $i = 1,2$ *and around* $\mathrm{x_n}$, for $\mathrm{i = n-1, n}$ *respectively, the local truncation errors* $t_i, i = 3,4,\dots,n-2$ *associated with the scheme* (16) *are*

$$\begin{aligned} t_i \ =\ & (2\omega+2\rho+\sigma)y_i+\left(-\omega-\rho-\frac{\sigma}{2}\right)h\,y_i^{(1)} +\left(\frac{34\omega+10\rho+\sigma}{8}-(2\alpha+2\beta+\gamma)\right)h^2 y_i^{(2)} \\ & +\left(\frac{-98\omega-26\rho-\sigma}{48}-(-\alpha-\beta-\frac{\gamma}{2})\right)h^3 y_i^{(3)} +\left(\frac{706\omega+82\rho+\sigma}{384}-\left(\frac{34\alpha+10\beta+\gamma}{8}\right)\right)h^4 y_i^{(4)} \\ & +\left(\frac{-2882\omega-242\rho-\sigma}{3840}-\left(\frac{-98\alpha-26\beta-\gamma}{48}\right)\right)h^5 y_i^{(5)} +\left(\frac{16354\omega+730\rho+\sigma}{46080}-(\frac{706\alpha+82\beta+\gamma}{384})\right)h^6 y_i^{(6)} \\ & +\left(\frac{-75938\omega-2186\rho-\sigma}{645120} -(\frac{-2882\alpha-242\beta-\gamma}{3840})\right)h^7 y_i^{(7)} \\ & +\left(\frac{397186\omega+6562\rho+\sigma}{(40320)(256)} -\left(\frac{16354\alpha+730\beta+\gamma}{46080}\right)\right)h^8 y_i^{(8)} \\ & +\mathrm{O}(\mathrm{h}^9),\quad \mathrm{i = 3,4,\dots,n-2} \quad (26) \end{aligned}$$

$$t_i \ =\ (a_0+\sigma_0+\sigma_1+\sigma_2+\sigma_3)y_i + \left(\frac{\sigma_0+3\sigma_1+5\sigma_2+7\sigma_3}{2}\right)h\,y_i^{(1)}$$

ISSN 2455-4863 (Online) www.ijisset.org Volume: 4 Issue: 9 | 2018

$$+\left(\begin{array}{c}\left(\frac{\sigma_0+9\sigma_1+25\sigma_2+49\sigma_3}{8}\right)- \\ (w_0+w_1+w_2+w_3+w_4+w_5+w_6)\end{array}\right)h^2y_i^{(2)}$$

$$+\left(\left(\frac{\sigma_0+27\sigma_1+125\sigma_2+343\sigma_3}{48}\right)\right.$$

$$\left.-\left(\frac{w_1+3w_2+5w_3+7w_4+9w_5+11w_6}{2}\right)\right)h^3y_i^{(3)}$$

$$+\left(\left(\frac{\sigma_0+81\sigma_1+625\sigma_2+2401\sigma_3}{384}\right)\right.$$

$$\left.-\left(\frac{w_1+9w_2+25w_3+49w_4+81w_5+121w_6}{8}\right)\right)h^4y_i^{(4)}$$

$$+\left(\left(\frac{\sigma_0+243\sigma_1+3125\sigma_2+16807\sigma_3}{3840}\right)\right.$$

$$\left.-\left(\frac{w_1+27w_2+125w_3+343w_4+729w_5+1331w_6}{48}\right)\right)h^5y_i^{(5)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_0+729\sigma_1+15625\sigma_2+117649\sigma_3}{46080}\right) \\ -\left(\frac{w_1+81w_2+625w_3+2401w_4+6561w_5+14641w_6}{384}\right)\end{array}\right)h^6y_i^{(6)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_0+2187\sigma_1+78125\sigma_2+823543\sigma_3}{645120}\right) \\ -\left(\frac{w_1+243w_2+3125w_3+16807w_4+59049w_5+161051w_6}{3840}\right)\end{array}\right)h^7y_i^{(7)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_0+6561\sigma_1+390625\sigma_2+5764801\sigma_3}{(256)(40320)}\right) \\ -\left(\frac{w_1+729w_2+15625w_3+117649w_4+531441w_5+1771561w_6}{46080}\right)\end{array}\right)h^8y_i^{(8)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_0+19683\sigma_1+1953125\sigma_2+40353607\sigma_3}{(512)(362880)}\right) \\ -\left(\frac{w_1+2187w_2+78125w_3+823543w_4+4782969w_5+19487171w_6}{645120}\right)\end{array}\right)h^9y_i^{(9)}$$

$$+O(h^{10}), i=1,n(27)$$

$$t_i = (b_0+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8)y_i + \left(\frac{\sigma_4+3\sigma_5+5\sigma_6+7\sigma_7+9\sigma_8}{2}\right) h\, y_i^{(1)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_4+9\sigma_5+25\sigma_6+49\sigma_7+81\sigma_8}{8}\right) \\ -(w_7+w_8+w_9+w_{10}+w_{11}+w_{12}+w_{13})\end{array}\right)h^2y_i^{(2)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_4+27\sigma_5+125\sigma_6+343\sigma_7+729\sigma_8}{48}\right)- \\ \left(\frac{w_7+3w_8+5w_9+7w_{10}+9w_{11}+11w_{12}+13w_{13}}{2}\right)\end{array}\right)h^3y_i^{(3)}$$

$$+\left(\left(\frac{\sigma_4+81\sigma_5+625\sigma_6+2401\sigma_7+6561\sigma_8}{384}\right)\right.$$

$$\left.-\left(\frac{w_7+9w_8+25w_9+49w_{10}+81w_{11}+121w_{12}+169w_{13}}{8}\right)\right)h^4y_i^{(4)}$$

$$+\left(\left(\frac{\sigma_4+243\sigma_5+3125\sigma_6+16807\sigma_7+59049\sigma_8}{3840}\right)\right.$$

$$-\left(\frac{w_7+27w_8+125w_9+343w_{10}+729w_{11}+13311w_{12}+2197w_1}{48}\right.$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_4+729\sigma_5+15625\sigma_6+117649\sigma_7+531441\sigma_8}{46080}\right) \\ -\left(\frac{w_7+81w_8+625w_9+2401w_{10}+6561w_{11}+14641w_{12}+28561}{384}\right.\end{array}\right.$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_4+2187\sigma_5+78125\sigma_6+823543\sigma_7+4782969\sigma_8}{645120}\right) \\ -\left(\frac{w_7+243w_8+3125w_9+16807w_{10}+59049w_{11}+161051w_{12}+371293w_{13}}{3840}\right)\end{array}\right)h^7y_i^{(7)}$$

$$+\left(\begin{array}{c}\left(\frac{\sigma_4+6561\sigma_5+390625\sigma_6+5764801\sigma_7+43046721\sigma_8}{(256)(40320)}\right) \\ -\left(\frac{w_7+729w_8+15625w_9+117649w_{10}+531441w_{11}+1771561w_{12}+4826809w_{13}}{46080}\right)\end{array}\right)h^8y_i^{(8)}$$

$$+O(h^{10}), i=2,n-1(28)$$

The scheme (16 – 4.20) gives rise to a high order accurate method as follows:

Six order convergent method

For $(\omega,\rho,\sigma)=(1,4,-10)$ *then* $(\alpha,\beta,\gamma)=\left(\frac{1}{20},\frac{22}{15},\frac{149}{30}\right)$

And

$(a_0,\sigma_0,\sigma_1,\sigma_2,\sigma_3)=$ $(30\,,-30\,,-30\,,$ *45 ,−15 w0,w1, w2, w3, w4, w5,w6= −194377616 , 29888328672 , 116939386016 , −16490914336 , −123199100352 ,−616125804832933153 92b0,σ4, σ5, σ6, σ7,σ8= −15 , −50 , 3152 ,−2552 ,50,−15*

$(w_7,w_8,w_9,w_{10},w_{11},w_{12},w_{13})$
$=\left(\begin{array}{c}-6789241\,,-43751194, 11647289,-12993516, 2538889, \\ -1136090, 164503\end{array}\right)/v$

Where $v=688128$ *, Then the local truncation errors are*

$$t_i=\begin{cases}\frac{-44335}{3612672}h^9y_i^{(9)}+O(h^{10})\ ;\ i=1,n \\ \frac{-2565113}{12386304}h^9y_i^{(9)}+O(h^{10})\ ;\ i=2,n-1 \\ \frac{1}{15120}h^8y_i^{(8)}+O(h^9)\ ;\ i=3,\ldots\ldots,n-2\end{cases}\quad(29)$$

Remark II

Choosing different values of the parameters (ω,ρ,σ) *along with* (α,β,γ)*. We get another scheme, for arbitrary choices of* $(\omega,\rho,\sigma)=(1,-1\,,0)$ *along with* $(\alpha,\beta,\gamma)=\left(\frac{17}{240},\frac{29}{30},\frac{37}{40}\right)$ *, gives* $t_i=\frac{-53}{20160}h^8y_i^{(8)}+O(h^9)$

2. METHOD OF ANALYSIS

The spline solution of (1) with the boundary condition (2) is based on the linear equations given by (16 – 4.20).Let $Y = \left(y_{i+1/2}\right), S = \left(s_{i+1/2}\right),$ $C = (c_i), T = (t_i), E = e_{i+1/2} = Y - S$ *is n-dimensional column vectors, and then we can write the standard matrix equations for the nonpolynomial spline method in the form:*

$$(i) NY = C + T$$

$(ii) NS = C (30) (iii) NE = T$

We also have $N = -\epsilon N_0 + h^2 BF, F = diag\left(f_{i+1/2}\right)$ (31)

Where

$$N = \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & & & & & \\ \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & & & & \\ \omega & \rho & \sigma & \rho & \omega & & & & \\ & & & & & & & & \\ & & & & \omega & \rho & \sigma & \rho & \omega \\ & & & & \sigma_8 & \sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 \\ & & & & & \sigma_3 & \sigma_2 & \sigma_1 & \sigma_0 \end{bmatrix} \quad (32)$$

The matrix B has the form

$$B = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & & & & & & & \\ w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} & w_{13} & & & & & & \\ \alpha & \beta & \gamma & \beta & \alpha & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & \alpha & \beta & \gamma & \beta & \alpha \\ & & & & & & w_{13} & w_{12} & w_{11} & w_{10} & w_9 & w_8 & w_7 \\ & & & & & & & w_6 & w_5 & w_4 & w_3 & w_2 & w_1 \end{bmatrix} \quad (33)$$

For the vector , we have

$$C_i = \begin{cases} \alpha_0(\varepsilon a_0 - w_0 f_0) + h^2\left[w_0 g_0 + \sum_{i=1}^{i=6} w_i g_{i-(1/2)}\right] & , i = 1 \\ \alpha_0 \varepsilon b_0 + h^2\left[\sum_{i=7}^{i=13} w_i g_{i-(13/2)}\right] & , i = 2 \\ h^2\left[\alpha\left(g_{i-5/2} + g_{i+3/2}\right) + \beta\left(g_{i-3/2} + g_{i+1/2}\right) + \gamma\left(g_{i-1/2}\right)\right], & i = 3, \ldots\ldots, n-2 \\ \alpha_1 \varepsilon b_0 + h^2\left[\sum_{i=7}^{i=13} w_i g_{n-i+(13/2)}\right], & i = n-1 \\ \alpha_1(\varepsilon a_0 - w_0 f_n) + h^2\left[w_0 g_n + \sum_{i=1}^{i=6} w_i g_{n-i+(1/2)}\right] & , i = n \end{cases} \quad (34)$$

2.1 CONVERGENCE ANALYSIS

Our main purpose now is to derive a bound on $\|E\|$ *.Now we turn back to the error equation (iii) in (30) and rewrite it in the form*

$$E = N^{-1} T \quad (35)$$

Let$M_* = min_{1 \le i \le n} \left|f_{i-1/2}\right|, M^* = max_{1 \le i \le n} \left|f_{i-1/2}\right|$; $0 < M_* \le M \le M^*$

Where M is some positive constant, let $n(i,j)$ *be the* (i,j) *element of the matrix N given by (31), clearly, the row sums* $S_1, S_2, \ldots\ldots, S_n$ *of the matrix N satisfy*

$$S_i = \sum_{j=1}^{j=n} n(i,j) \ge -\epsilon(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3) + h^2(w_1 + w_2 + w_3 + w_4 + w_5 + w_6)M_* \quad , for \quad i = 1, n \quad (36)$$

$$S_i = \sum_{j=1}^{j=n} n(i,j) \ge -\epsilon(\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8) + h^2(w_7 + w_8 + w_9 + w_{10} + w_{11} + w_{12} + w13M* \quad ,for\ i=2, n-1 (37)$$

$$S_i = \sum_{j=1}^{j=n} n(i,j) \ge h^2(2(\alpha+\beta)+\gamma)M_* \ , \ for\ i = 3, \ldots\ldots, n-2 (38)$$

Since $0 < \epsilon \ll 1$ *, if we choose h to be* $O(\sqrt{\epsilon})$ *then N becomes irreducible and monotone (Henrici,1962). Hence* N^{-1} *exist and its elements are nonnegative, so*

$$\sum_{i=1}^{i=n} \left(n^{-1}_{(j,i)}\ S_i\right) = 1\ ,\ j = 1,2,\ldots,n (39)$$

Where $n^{-1}_{(j,i)}$ *is the* (j,i) *element of the matrix* N^{-1}

Therefore

$$\sum_{i=1}^{i=n} n^{-1}_{(j,i)} \le \frac{1}{min_{1 \le i \le n} S_i} (40)$$

Since, $e_j = \sum_{i=1}^{i=n} \left(n^{-1}_{(j,i)} . t_i\right),\ j = 1,2,\ldots,n \quad (41)$

Since,$\|E\| = max_{1 \le j \le n} \left|e_j\right|$ *and*

$$\|T\| = \frac{1}{15120} h^8 M_8, where M_8 = max_{a \le x \le b} \left|y(x)^{(8)}\right|, \quad (42)$$

$$Then \quad \|E\| \cong O(h^6) \quad (43)$$

We summarize the above results in the next theorem

Theorem: Let $y(x)$ *be the exact solution of the continuous boundary value problem (1) with the boundary condition (2) and let*$y_{i+1/2}$, $i = 0, 1, \ldots, n-1$ *satisfy the discrete boundary value problem (ii) in (30).*

Further, if $e_{i+1/2} = y_{i+1/2} - S_{i+1/2}$ *then* $\|E\| \cong O\ (h^6)$*, is a six order method which is given by* (43).

2.2 Numerical Examples

We now consider two numerical examples to illustrate the comparative performance of our method (ii) in (30) over other existing methods. All calculations are implemented by MATLAB 7

Example 1. Consider the boundary value problem [3]

$$-\epsilon y^{(2)} + (1+x)y = -40(x^3 - x - 2\epsilon)$$

$$y(0) = y(1) = 0(44)$$

The analytical solution of (44) is

$$y(x) = 40x(1-x) \qquad (45)$$

This problem has been solved by using our six order method given by equation (29) with different values of $= 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$ *at* $n = 32$*, the maximum absolute errors are tabulated in Table (1).*

Example 2. Consider the boundary value problem [14]

$$-\epsilon y^{(2)} + y = -cos^2(\pi x) - 2\,\epsilon\,\pi^2 cos(2\pi x)$$

$$y(0) = y(1) = 0(46)$$

The analytical solution of (46) is

$$y(x) = \frac{e^{\frac{x-1}{\sqrt{\epsilon}}} + e^{\frac{-x}{\sqrt{\epsilon}}}}{1 + e^{\frac{-1}{\sqrt{\epsilon}}}} - cos^2(\pi x) \qquad (47)$$

This problem has been solved by our six order method given by equation (29) with different values of the parameters ϵ *and* n *.In these tables we have compared our results with the methods in [5, 7-9, 11, 17-19], the maximum absolute errors are tabulated in tables (2 - 10).*

ϵ	Miller [16]	Niijima [3]	Niijima [15]	Our six order method
10*^-4	6.4*10^-3	6.5*10^-3	5.9*10^-5	4.44*10^-15
10*^-5	6.1*10^-3	6.4*10^-3	2.1*10^-5	3.55*10^-15
10*^-6	4.1*10^-3	5.6*10^-3	3.5*10^-5	3.55*10^-15
10*^-7	7.7*10^-4	3.1*10^-3	3.9*10^-5	5.33*10^-15
10*^-8	7.6*10^-5	1.2*10^-3	2.1*10^-5	5.33*10^-15

Table (1): The observed maximum absolute errors for Example 1

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$
$\frac{1}{16}$	1.42*10^-7	2.06*10^-10	3.36*10^-12	5.85*10^-14
$\frac{1}{32}$	9.56*10^-8	1.72*10^-10	2.92*10^-12	4.66*10^-14
$\frac{1}{64}$	4.29*10^-7	2.30*10^-9	9.36*10^-12	8.74*10^-14
$\frac{1}{128}$	6.99*10^-6	3.56*10^-8	1.59*10^-10	8.56*10^-13

Table (2): The observed maximum absolute errors for Example 2

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	3.53*10^-7	4.49*10^-9	6.92*10^-11	1.07*10^-12
$\frac{1}{32}$	2.62*10^-7	3.39*10^-9	5.85*10^-11	9.20*10^-13
$\frac{1}{64}$	5.34*10^-7	5.55*10^-9	1.04*10^-10	1.68*10^-12
$\frac{1}{128}$	9.27*10^-6	3.30*10^-8	8.34*10^-10	1.42*10^-11

Table (3): The observed maximum absolute errors for Example 2 in [11]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	1.22*10^-6	6.45*10^-9	3.40*10^-11	1.03*10^-12
$\frac{1}{32}$	2.00*10^-6	1.68*10^-8	1.36*10^-10	1.09*10^-12
$\frac{1}{64}$	8.89*10^-6	1.16*10^-7	1.20*10^-9	1.08*10^-11
$\frac{1}{128}$	5.74*10^-5	9.98*10^-7	1.18*10^-8	1.14*10^-10

Table (4): The observed maximum absolute errors for Example 2 [8]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	8.06*10^-3	2.02*10^-3	5.08*10^-4	1.27*10^-4
$\frac{1}{32}$	7.11*10^-3	1.79*10^-3	4.48*10^-4	1.12*10^-4
$\frac{1}{64}$	6.58*10^-3	1.66*10^-3	4.15*10^-4	1.04*10^-4
$\frac{1}{128}$	6.36*10^-3	1.61*10^-3	4.03*10^-4	1.01*10^-4

Table (7): The observed maximum absolute errors for Example 2 in [5]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	7.09*10^-3	1.77*10^-3	4.45*10^-4	1.11*10^-4
$\frac{1}{32}$	5.68*10^-3	1.42*10^-3	3.55*10^-4	8.89*10^-5
$\frac{1}{64}$	4.07*10^-3	1.01*10^-3	2.54*10^-4	6.35*10^-5
$\frac{1}{128}$	6.97*10^-3	1.75*10^-3	4.33*10^-4	1.08*10^-4

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 4 Issue: 9 | 2018*

Table (10): The observed maximum absolute errors for Example 2 in [19]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	4.07*10^-5	2.53*10^-6	1.58*10^-7	9.87*10^-9
$\frac{1}{32}$	2.00*10^-5	1.24*10^-6	7.74*10^-8	4.83*10^-9
$\frac{1}{64}$	5.45*10^-5	3.42*10^-6	2.14*10^-7	1.34*10^-8
$\frac{1}{128}$	1.83*10^-4	1.22*10^-5	7.68*10^-7	4.81*10^-8

Table (6): The observed maximum absolute errors for Example 2 in [9]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	1.20*10^-4	7.47*10^-6	4.67*10^-7	2.90*10^-8
$\frac{1}{32}$	1.28*10^-4	8.00*10^-6	5.00*10^-7	3.14*10^-8

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	4.14*10^-3	1.02*10^-3	2.54*10^-4	6.35*10^-5
$\frac{1}{32}$	3.68*10^-3	9.03*10^-4	5.01*10^-5	1.40*10^-5
$\frac{1}{64}$	3.45*10^-3	8.40*10^-3	2.08*10^-4	5.20*10^-5
$\frac{1}{128}$	3.45*10^-3	8.21*10^-4	2.03*10^-4	5.06*10^-5

Table (8): The observed maximum absolute errors for Example 2 in [17]

$\frac{1}{64}$	1.60*10^-4	1.00*10^-6	6.26*10^-7	3.92*10^-8
$\frac{1}{128}$	2.34*10^-4	1.47*10^-6	9.23*10^-7	5.77*10^-8

Table (9): The observed maximum absolute errors for Example 2 in [18]

ϵ	n = 16	n = 32	n = 64	n = 128
$\frac{1}{16}$	1.57*10^-5	8.79*10^-7	5.32*10^-8	3.30*10^-9
$\frac{1}{32}$	8.27*10^-6	4.41*10^-7	2.62*10^-8	1.62*10^-9
$\frac{1}{64}$	1.84-*10^5	8.67*10^-7	6.65*10^-8	4.39*10^-9
$\frac{1}{128}$	1.03*10^-4	2.61*10^-6	2.23*10^-7	1.54*10^-8

Table (5): The observed maximum absolute errors for Example 2 in [7]

3. CONCLUSIONS

We have described a numerical method for solving singularly perturbed second order two point boundary value problem using quartic nonpolynomial spline at midknotes. It is a computationally efficient method; its algorithm can easily be implemented on a computer and needless coefficients evaluations as compared with quintic and sextic splines [8, 11]. *The tables confirm theoretical convergence and demonstrate the superiority of the proposed method against the existing methods.*

REFERENCES

[1] *M.K.Kadalbajoo,K.C.Patidar,"A survey of numerical techniques for solving singularly perturbed ordinary differential equations", Appl. Math. Comp. Vol.130 PP. 457-510,2007*

[2] *A.F.Hegarty,J.J.H.Miller,E.O'Riordan,"Uniform second order difference schemes for singular perturbationproblems", :J.J.H.Miller(Ed.),Boundary and Interior layers-Computational and asymptotic methods, Boole Press,Dublin,,PP.301-305,1980.*

[3] *K.Niijima,"On A three point difference scheme for a singular perturbation problem without a first derivative term",I,Mem.Numer.Math. Vol7,PP. 1-10 ,1980.*

[4] *I.P.Boglaev,"A vartional difference scheme for a boundary value problem with a small parameter inthehighestderivative",U.S.S.R.,Comput.Math.phys.Vol .21(4),PP. 71-81 ,1981.*

[5] *K.Surla,M.Stojanovic,"Solving singularly perturbed boundary value problems by splines in tensions" ,Comp.Appl.Math.Vol.44,PP187-199,1992.*

[6] *R.K.Mohanty,NavnitJha , "A class of variable mesh spline in compression methods for singularly perturbed two point singular boundary value problems",Appl.Math.Comp.Vol.168,PP.704-716,2005.*

[7] *T.Aziz,A.Khan,"Quintic spline approach to the solution of a singularly perturbed boundary value problems",J.Optim.Theoryand Appl.Vol.112,PP517-527,2002.*

[8] *A.Khan, I.Khan, T.Aziz,"Sextic spline solution of a singularly perturbed boundary value problems", Appl.Math.Comp.Vol.181,PP.432-439,2006.*

[9] *J.Rashidinia, M.Ghasemi, Z.Mahamoodi,"Spline approach to the solution of a singularly perturbed boundary value problems", Appl.Math.Comput.Vol.189,PP.72-78,2007.*

ISSN 2455-4863 (Online) **www.ijisset.org** *Volume: 4 Issue: 9 | 2018*

[10] *M.K.Kadalbajoo and V.K.Aggarwal, "Fitted Mesh B-spline collocation method for solving self adjoint singularly perturbed boundary value problems", Appl.Math.Comput.Vol.161,PP.973-987,2005.*

[11] *M.A.Ramadan,I.F.Lashien and W.K.Zahra,"The numerical solution of singularly perturbed boundary value problems using nonpolynomial spline",InternationalJ.Pureand Appl.Math.Vol.41,PP.883-896,2007.*

[12] *E.A. Al-Said, "Quartic Splines solutions for a system of second order boundary value problems" ,Comp. Appl.Math.Vol.166,PP.254-264,2005.*

[13] *R.A. Usmani,"The use of Quartic Splines in the numerical solutionOf a fourth order boundary value problems", Comp.Appl.Math.Vol.44,PP.187-199,1992.*

[14] *P.,Henrici,"Discrete variables methods in ordinary differential equations", John Wiley and Sons, New York ,NY,1962.*

[15] *K.Niijima,"On a three point difference scheme for a singular perturbation problem without a first derivatives term II" ,Mem.Num.Math.Vol.7,PP.11-27,1980.*

[16] *J.J.H.Miller,"On the convergence,Uniformly in* ϵ *of difference schemes for a two point boundary singular perturbation problem ,in:Numerical analysis of singular perturbation problems" ,Academic press,New York ,PP.467-474,1979.*

[17] *K.Surla,D.Herceg and L.Cvetkovic,"A finally of exponential spline difference scheme" ,Rev.Res.,Math.Ser.,Univ.Novi Sad ,Vol.19,PP.12-23,1991.*

[18] *K.Surla,V.Vukoslavcevic, "A spline difference scheme for boundary value problems with a small parameter",Rev.Res.,Math.Ser.,Univ.Novi Sad Vol.25,PP.159-166,1995.*

[19] *M.K.Kadalbajoo and R.K.Bawa,"Variable mesh difference scheme for singularly perturbed boundary value problem using splines", J.Optim.Theory and Appl.PP.405-416,1996*

AUTHORS' BIOGRAPHIES

Dr.Zaki Ahmed Zaki, Lecturer of Engineering Mathematics, Engineering Mathematics and Physics Department, Faculty of Engineering (shoubra), Benha University, Cairo, 11629, Egypt.

Specific Specialization: Numerical Analysis –Differential Equations.