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Eigenfunction Expansion for Sturm-Liouville Problems

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Abstract: *In this paper, the boundary value problem for the Sturm-Liouville equation with discontinuity conditions inside a finite interval is considered. The completeness theorem for eigenfunctions of this problem is proved. The spectral expansion formula with respect to eigenfunctions is obtained by using the contour integral method and Parseval equality is given.*

Keywords: *Boundary value problem, Strum-Liouville equation, discontinuity conditions, completeness theorem, expansion formula.*

1. INTRODUCTION

In mathematical physics, geophysics, electromagnetic, elasticity and other branches of engineering and natural sciences, the boundary value problems for Sturm-Liouville equation with discontinuity inside an interval are often encountered [1-6]. Such problems are connected with discontinuous material properties (see [7] for details).

Now, in this work, we consider the Sturm-Liouville equation

$$-y''+q(x)y=\lambda y, \qquad 0<x<\pi, \tag{1}$$

with the boundary condition

$$y'(0)=y(\pi)=0, \tag{2}$$

and the discontinuity conditions

$$y(d+0)=ay(d-0), \quad y'(d+0)=a^{-1}y'(d-0) \tag{3}$$

where $q(x)\in L_2(0,\pi)$ is a real valued function, $d\in\left(\frac{\pi}{2},\pi\right)$, a is real number and $0<a\neq 1$, λ is a spectral parameter.

In continuous case, i.e. $a=1$, direct and inverse spectral problems of Sturm-Liouville operators are excessively examined, especially the books by V. A. Marchenko [8] and by B. M. Levitan and I. S. Sargsjan [9] may serve as a good introduction to the theory of Sturm-Liouville operators. In the recent years, the Sturm-Liouville problems which has discontinuities in the solution or its derivative at interior point has been widely studied at different aspects [7, 10-17]. The aim of this work is the prove that the system of the eigenfunctions of the Sturm-Liouville boundary value problem with discontinuity (1)-(3) is complete and forms an orthogonal basis in $L_2(0,\pi)$. This paper is organized as follows: in section 2, we give some spectral properties of the problem (1)-(3) for preliminaries. Namely, the asymptotic formulas of the eigenvalues, eigenfunctions and normalizing numbers of the boundary value problem (1)-(3) are given. In section 3, we prove the completeness theorem of the problem (1)-(3) and obtain the spectral expansion formula with respect to the eigenfunctions of this problem. Note that the completeness and expansion theorems are important for solving various problems in mathematical physics by the Fourier method.

2. PRELIMINARIES

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solution of equation (1) satisfying the following initial conditions

$$\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=0$$
$$\psi(\pi,\lambda)=0, \qquad \psi'(\pi,\lambda)=1$$

and the discontinuity condition (3). Let $\lambda=k^2$. The integral representation of solution $\varphi(x,\lambda)$ has the form ([10])

$$\varphi(x,\lambda)=\varphi_0(x,k)+\int_0^x \tilde{K}(x,t)\cos kt\,dt, \tag{4}$$

where

$$\varphi_0(x,k)=\begin{cases}\cos kx, & 0<x<d,\\ a^+\cos kx+a^-\cos k(2d-x), & d<x<\pi,\end{cases}$$

here, $a^\pm=\frac{1}{2}\left(a\pm\frac{1}{a}\right)$, the kernel

$$\tilde{K}(x,t)=K(x,t)-K(x,-t)$$

and

$$\int_{-x}^{x}|K(x,t)|dt\le e^{c\sigma_1(x)}-1, \qquad \sigma_1(x)=\int_0^x (x-t)|q(t)|dt.$$

The integral representation (4) is not a transformation operator. Moreover, differently from the transformation operator, the kernel function of this representation has a discontinuity along the line $t=2d-x, \quad x>d$.

The characteristic function of the boundary value problem (1)-(3) is

$\Delta(\lambda)=\langle\psi(x,\lambda),\varphi(x,\lambda)\rangle=\psi(x,\lambda)\varphi'(x,\lambda)-\psi'(x,\lambda)\varphi(x,\lambda)$, and $\langle\psi(x,\lambda),\varphi(x,\lambda)\rangle$ is not depend on x. Substituting $x=0$ and $x=\pi$ in this relation

$$\Delta(\lambda)=-\varphi(\pi,\lambda)=-\psi'(0,\lambda).$$

The zeros λ_n of characteristic function coincide with the eigenvalues of the boundary value problem (1)-(3). The function $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions and there exists a sequence β_n such that

$$\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n), \qquad \beta_n\neq 0. \tag{5}$$

The eigenvalues of the problem (1)-(3) are simple and

$$\dot{\Delta}(\lambda_n)=-\alpha_n\beta_n, \tag{6}$$

holds, where

$$\alpha_n:=\int_0^\pi \varphi^2(x,\lambda_n)dx$$

is the normalizing numbers of problem (1)-(3).

Lemma 1. [10]

i. *The eigenvalues of problem (1-3) have the following asymptotic behavior*

$$\sqrt{\lambda_n}=k_n=k_n^0+\frac{d_n}{k_n^0}+\frac{\delta_n}{k_n^0}, \tag{7}$$

where d_n is a bounded sequence and $\{\delta_n\}\in l_2$.

ii. *Normalizing numbers of the problem (1)-(3) have the asymptotic behavior*

$$\alpha_n=\alpha_n^0+\tilde{\delta}_n, \qquad \{\tilde{\delta}_n\}\in l_2.$$

In this lemma, $\left(k_n^0\right)^2$ and α_n^0 are eigenvalues and normalizing numbers respectively in case of $q(x)\equiv 0$ in the equation (1).

Lemma 2. *The eigenfunctions of the boundary value problem have the form*

$$\varphi(x,\lambda_n)=\varphi_0(x,k_n^0)+\frac{\xi_n(x)}{k_n^0}, \qquad |\xi_n(x)|\le C. \tag{8}$$

Proof. Substituting the eigenvalues (7) into the expression (4), we get for $x<d$,

$$\varphi(x,\lambda_n)=\cos k_n^0x+\frac{\hat{\xi}_n(x)}{k_n^0},$$

where

$$\hat{\xi}_n(x)=\tilde{K}(x,x)\left[\sin k_n^0x+\frac{(d_n+\delta_n)x}{k_n^0}\cos k_n^0x\right]$$

$$-\int_0^x\tilde{K}'(x,t)\sin k_n^0t\,dt-\frac{(d_n+\delta_n)}{k_n^0}\int_0^x\tilde{K}'(x,t)t\cos k_n^0t\,dt.$$

Similarly, for $x>d$, we have

$$\varphi(x,\lambda_n)=a^+\cos k_n^0x+a^-\cos k_n^0(2d-x)+\frac{\breve{\xi}_n(x)}{k_n^0},$$

where

$$\breve{\xi}_n(x)=\tilde{K}(x,x)\left[\sin k_n^0x+\frac{(d_n+\delta_n)x}{k_n^0}\cos k_n^0x\right]$$

$$-\left[\tilde{K}(x,t)\Big|_{t=2d-x-0}^{t=2d-x+0}\right]\left[\sin k_n^0(2d-x)+\frac{(d_n+\delta_n)}{k_n^0}(2d-x)\cos k_n^0(2d-x)\right]$$

$$-\int_0^x\tilde{K}'(x,t)\sin k_n^0t\,dt-\frac{(d_n+\delta_n)}{k_n^0}\int_0^x\tilde{K}'(x,t)t\cos k_n^0t\,dt.$$

Consequently, using the relation (see [10])

$$\tilde{K}(x,x)=\frac{a^+}{2}\int_0^x q(t)dt,$$

$$\tilde{K}(x,t)\Big|_{t=2d-x-0}^{t=2d-x+0}=\frac{a^-}{2}\int_0^x q(t)dt,$$

$|\hat{\xi}_n(x)|\le c_1$ and $|\breve{\xi}_n(x)|\le c_2$. Thus we obtain the formula (8). The lemma is proved.

3. MAIN RESULTS

The completeness and expansion theorems are important for solving various problems in mathematical physics by the Fourier method, and also for the spectral theory itself. In this section, we prove the completeness theorem of the eigenfunctions of the problem (1)-(3) and then obtain the expansion formula with respect to eigenfunctions of this problem.

Theorem 3. *The system of eigenfunctions $\{\varphi(x,\lambda_n)\}_{n\ge 0}$ of the boundary value problem (1)-(3) is complete in $L_2(0,\pi)$.*

Proof. Denote

$$G(x,t,\lambda)=-\frac{1}{\Delta(\lambda)}\begin{cases}\psi(x,\lambda)\varphi(t,\lambda), & x\ge t,\\ \varphi(x,\lambda)\psi(t,\lambda), & x\le t,\end{cases} \quad (9)$$

and consider the function

$$Y(x,\lambda)=\int_0^\pi G(x,t,\lambda)f(t)dt. \quad (10)$$

The function $Y(x,\lambda)$ is the solution of the equation

$$\begin{gathered}-Y''+q(x)Y=\lambda Y+f(x),\\ Y'(0)=Y(\pi)=0,\\ Y(d+0)=aY(d-0),\\ Y'(d+0)=a^{-1}Y'(d-0).\end{gathered} \quad (11)$$

The function $G(x,t,\lambda)$ is called Green function of the problem (1)-(3). It is written from (9) and (10) that

$$Y(x,\lambda)=-\frac{\psi(x,\lambda)}{\Delta(\lambda)}\int_0^x\varphi(t,\lambda)f(t)dt-\frac{\varphi(x,\lambda)}{\Delta(\lambda)}\int_x^\pi\psi(t,\lambda)f(t)dt.$$

It follows from (5) and (6) that

$$\psi(x,\lambda_n)=-\frac{\dot{\Delta}(\lambda_n)}{\alpha_n}\varphi(x,\lambda_n).$$

Using these expressions, we get

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=\frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^\pi\varphi(t,\lambda_n)f(t)dt\,. \quad (12)$$

Let $f(x)\in L_2(0,\pi)$ be such that

$$f(x)=\int_0^\pi f(t)\varphi(t,\lambda_n)dt=0, \qquad n\ge 0.$$

Then, it follows from (12) that $\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=0.$ Consequently, for each fixed $x\in(0,\pi)$, $Y(x,\lambda)$ is a entire function with respect to λ. Moreover, in the equality (10), taking into account the inequality (see [10])

$$|\Delta(\lambda)|\ge C_\delta|k|e^{|\operatorname{Im}k|\pi} \quad (13)$$

which is valid in the domain

$$G_\delta:=\left\{k:\left|k-k_n^0\right|\ge\delta,\ \ n\ge 0\right\}$$

where δ is sufficiently small positive number and the asymptotic formulas

$$\varphi(x,\lambda)=O\left(e^{|\operatorname{Im}k|x}\right),\quad |k|\to\infty, \quad (14)$$

$$\psi(x,\lambda)=O\left(\frac{e^{|\operatorname{Im}k|(\pi-x)}}{|k|}\right),\quad |k|\to\infty \quad (15)$$

it is obtained that for fixed $\delta>0$ and sufficiently large $k^*>0$:

$$|Y(x,\lambda)|\le\frac{C_\delta}{|k|},\qquad k\in G_\delta,\ \ |k|\ge k^*.$$

Using maximum principle and Liouville theorem, we find $Y(x,\lambda)\equiv 0.$ It follows from here and (11) that $f(x)=0$ a.e. on $(0,\pi)$. The theorem is proved.

Theorem 4. *Let $f(x)$ be absolutely continuous function, $f(d+0)=af(d-0)$, $f'(d+0)=a^{-1}f'(d-0)$ and $f'(0)=f(\pi)=0$. Then, the expansion formula is valid:*

$$f(x)=\sum_{n=0}^\infty c_n\varphi(x,\lambda_n), \quad (16)$$

where

$$c_n=\frac{1}{\alpha_n}\int_0^\pi f(t)\varphi(t,\lambda_n)dt$$

and the series converges uniformly on $[0,\pi]$.

Proof. Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solution of the problem (1)-(3), we can write

$$\begin{aligned}Y(x,\lambda)=-\frac{1}{\lambda\Delta(\lambda)}\Big\{&\psi(x,\lambda)\int_0^x\left[-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda)\right]f(t)dt\\ &+\varphi(x,\lambda)\int_x^\pi\left[-\psi''(t,\lambda)+q(t)\psi(t,\lambda)\right]f(t)dt\Big\}.\end{aligned}$$

Integrating of the terms containing second derivatives by parts, we get for $x<d$

$$\begin{aligned}Y(x,\lambda)=-\frac{1}{\lambda\Delta(\lambda)}\Big\{&-\psi(x,\lambda)\varphi'(t,\lambda)f(t)\Big|_{t=0}^{x}\\ &+\psi(x,\lambda)\int_0^x\left[\varphi'(t,\lambda)f'(t)+q(t)\varphi(t,\lambda)f(t)\right]dt\\ &-\varphi(x,\lambda)\left[\psi'(t,\lambda)f(t)\left(\Big|_{t=x}^{d-0}+\Big|_{t=d+0}^{\pi}\right)\right]\\ &+\varphi(x,\lambda)\int_x^\pi\left[\psi'(t,\lambda)f'(t)+q(t)\psi(t,\lambda)f(t)\right]dt\Big\}\end{aligned}$$

and for $x>d$

$$Y(x,\lambda)=-\frac{1}{\lambda\Delta(\lambda)}\Big\{-\psi(x,\lambda)\varphi'(t,\lambda)f(t)\Big(\Big|_{t=0}^{d-0}+\Big|_{t=d+0}^{x}\Big)$$

$$+\psi(x,\lambda)\int_0^x\big[\varphi'(t,\lambda)f'(t)+q(t)\varphi(t,\lambda)f(t)\big]dt$$

$$-\varphi(x,\lambda)\psi'(t,\lambda)f(t)\Big|_{t=x}^{\pi}$$

$$+\varphi(x,\lambda)\int_x^\pi\big[\psi'(t,\lambda)f'(t)+q(t)\psi(t,\lambda)f(t)\big]dt\Big\}.$$

Thus, we obtain

$$Y(x,\lambda)=\frac{f(x)}{\lambda}-\frac{1}{\lambda}\big(Z_1(x,\lambda)+Z_2(x,\lambda)\big), \qquad (17)$$

where

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\int_0^x\varphi'(t,\lambda)g(t)dt$$

$$+\varphi(x,\lambda)\int_x^\pi\psi'(t,\lambda)g(t)dt\Big\},\quad g(t):=f'(t),$$

$$Z_2(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\int_0^x q(t)\varphi(t,\lambda)f(t)dt$$

$$+\varphi(x,\lambda)\int_x^\pi q(t)\psi(t,\lambda)f(t)dt\Big\}.$$

Using the expressions (13)-(15), for fixed $\delta>0$ and sufficiently large $k^*>0$, we have

$$\max_{0\le x\le\pi}\big|Z_2(x,\lambda)\big|\le\frac{C_2}{|k|},\quad k\in G_\delta,\quad |k|\ge k^*. \qquad (18)$$

Now, let us show that

$$\lim_{\substack{|k|\to\infty\\ k\in G_\delta}}\max_{0\le x\le\pi}\big|Z_1(x,\lambda)\big|=0. \qquad (19)$$

Assume that $g(x)$ is absolutely continuous on $[0,\pi]$. Then, integrating by parts, we find for $x<d$

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\varphi(t,\lambda)g(t)\Big|_{t=0}^{x}+$$

$$+\varphi(x,\lambda)\psi(t,\lambda)g(t)\Big(\Big|_{t=x}^{d-0}+\Big|_{t=d+0}^{\pi}\Big)$$

$$-\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g'(t)dt-\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g'(t)dt\Big\}$$

and for $x>d$

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\varphi(t,\lambda)g(t)\Big(\Big|_{t=0}^{d-0}+\Big|_{t=d+0}^{x}\Big)+$$

$$+\varphi(x,\lambda)\psi(t,\lambda)g(t)\Big|_{t=x}^{\pi}$$

$$-\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g'(t)dt-\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g'(t)dt\Big\}.$$

Thus, for both cases, we can write

$$Z_1(x,\lambda)=\frac{-1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g'(t)dt+\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g'(t)dt\Big\}.$$

Similarly, using the relations (13)-(15), we get

$$\max_{0\le x\le\pi}\big|Z_1(x,\lambda)\big|\le\frac{C_1}{|k|},\quad k\in G_\delta,\quad |k|\ge k^*.$$

In general case, fix $\varepsilon>0$ and choose absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi\big|g(t)-g_\varepsilon(t)\big|dt<\varepsilon.$$

Then, in the relation

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\int_0^x\big(g(t)-g_\varepsilon(t)\big)\varphi'(t,\lambda)dt$$

$$+\varphi(x,\lambda)\int_x^\pi\big(g(t)-g_\varepsilon(t)\big)\psi'(t,\lambda)dt\Big\}$$

$$-\frac{1}{\Delta(\lambda)}\Big\{\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g'_\varepsilon(t)dt+\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g'_\varepsilon(t)dt\Big\},$$

using the estimates (13)-(15), we calculate for $k^{**}>0$, $k\in G_\delta$, $|k|\ge k^{**}$,

$$\max_{0\le x\le\pi}\big|Z_1(x,\lambda)\big|\le C\int_0^\pi\big|g(t)-g_\varepsilon(t)\big|dt+\frac{\tilde{C}(\varepsilon)}{|k|}<C\varepsilon+\frac{\tilde{C}(\varepsilon)}{|k|}.$$

Thus, it follows from here that

$$\overline{\lim_{\substack{k\to\infty\\ k\in G_\delta}}}\max_{0\le x\le\pi}\big|Z_1(x,\lambda)\big|\le C\varepsilon.$$

Since ε is an arbitrary positive number, the relation (19) holds. Now, consider the following contour integral:

$$I_N(x,t)=\frac{1}{2\pi i}\int_{G_N}Y(x,\lambda)d\lambda,$$

where $G_N:=\Big\{k:\big|k_N^0\big|+\frac{\beta}{2},\quad N=0,1,...\Big\}$ is a contour with the counter-clockwise circuit. It is obtained from the residue theorem that

$$I_N(x,t)=\sum_{n=0}^{N}\operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda)=\sum_{n=0}^{N} c_n\varphi(x,\lambda_n), \qquad (20)$$

where

$$c_n=\frac{1}{\alpha_n}\int_0^{\pi} f(t)\varphi(x,\lambda_n)dt.$$

On the other hand, using the equality (17), we get

$$I_N(x,t)=f(x)-\varepsilon_N(x), \qquad (21)$$

where

$$\varepsilon_N(x)=\frac{1}{2\pi i}\int_{G_N}\frac{1}{\lambda}\left[Z_1(x,\lambda)+Z_2(x,\lambda)\right]d\lambda$$

and from (18), (19)

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\varepsilon_N(x)\right|=0.$$

Consequently, using the expressions (20) and (21), as $N\to\infty$, the expansion formula (16) is obtained. The theorem is proved.

Corollary 5. *The system of eigenfunctions* $\{\varphi(x,\lambda_n)\}_{n\ge 0}$ *is complete and orthogonal in* $L_2(0,\pi)$, *therefore it forms an orthogonal basis in* $L_2(0,\pi)$. *For* $f(x)\in L_2(0,\pi)$, *the Parseval equality is valid:*

$$\int_0^{\pi}\left|f(t)\right|^2 dt=\sum_{n=0}^{\infty}\alpha_n\left|c_n\right|^2.$$

REFERENCES

[1] R. S. Anderssen, "The effect of discontinuous in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequencies of the earth," *Geophys J. R. Astr. Soc.,* vol. 50, pp. 303-309, 1997.

[2] O. H. Hald, "Discontinuous inverse eigenvalue problems," *Comm. Pure Appl. Math.,* vol. 37, pp. 539-577, 1984.

[3] R. J. Kruger, "Inverse problems for nonabsorbing media with discontinuous material properties," *J. Math. Phys.,* vol. 23, pp. 396-404, 1982.

[4] F. R. Lapwood and T. Usami, *Free Oscillations of the Earth.* Cambridge: Cambridge Univ. Press, 1981.

[5] D. G. Shepelsky, "The inverse problem of reconstruction of medium's conductivity in a class of discontinuous and increasing functions," *Adv. Soviet Math.*, vol. 19, pp. 209-231, 1994.

[6] C. Willis, "Inverse problems for torsional modes," *Geophysical Journal of Royal Astronomical Society,* vol. 62, pp. 41-48, 1980.

[7] G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville Problems and Their Applications.* Huntington, N.Y.: Nova Science Publishers, Inc., 2001.

[8] V. A. Marchenko, *Sturm-Liouville Operators and Applications,* 2nd ed. Providence, Rhode Island: AMS Chelsea Publishing, 2011.

[9] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators.* Dordrecht, Boston, London: Kluwer Academic Publishers, 1991.

[10] R. Kh. Amirov, "On Strum-Liouville operators with discontinuity conditions inside an interval," *J. Math. Anal. Appl.,* vol. 317, pp. 163-176, 2006.

[11] M. Kadakal and O. Sh. Mukhtarov, "Discontinuous Sturm-Liouville problems containing eigenparameter in the boundary conditions," *Acta Mathematica Sinica, English Series,* vol. 22, pp. 1519-1528, 2006.

[12] O. Sh. Mukhtarov and K. Aydemir, "Eiegnfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point," *Acta Mathematica Scientia,* vol. 35, pp. 639-649, 2015.

[13] E. Şen, "A class of second-order differential operators with eigenparameter-dependent boundary and transmission conditions," *Mathematical Methods in the Applied Sciences,* vol. 37, pp. 2952-2961, 2014.

[14] E. Şen and O. Sh. Mukhtarov, "Spectral properties of discontinuous Sturm-Liouville problems with a finite number of transmission conditions," *Mediterr. J. Math.,* DOI 10.1007/s00009-014-0487-x, 2014.

[15] Y.-P. Wang and H. Koyunbakan, "On the Hochstadt-Lieberman theorem for discontinuous boundary-valued problems," *Acta Mathematica Sinica, English Series,* vol. 30, pp. 985-992, 2014.

[16] Q. Yang and W. Wang, "Asymptotic behavior of a differential operator with discontinuities at two points," *Mathematical Methods in the Applied Science,* vol. 34, pp. 373-383, 2011.

[17] C.-F. Yang, "An interior spectral problem for discontinuous boundary value problems," *Integral Equations and Operator Theory,* vol. 65, 593-604, 2009.