International Journal of Innovative Studies in Sciences and Engineering Technology (IJISSET)

ISSN 2455-4863 (Online) www.ijisset.org *Volume: 2 Issue: 12 | December 2016*

Global Stability of a Stochastic Prey-Predator Model with Time-Dependent Delays

Zepeng Xiong[a], Xiaoping Li[a,*], Xiangjun Dai[b]

[a]*Science college , Hunan Agriculture University, Changsha, 410128,PR. China*

[b]*School of Data Science of Tongren University, Tongren, 554300, PR China*

Abstract: An autonomous stochastic predator-prey system with time-dependent delays are proposed and studied. Sufficient conditions for global stability of the Positive equilibrium state are established, the obtained result shows that time-dependent delays have effects on the global stability of the positive equilibrium state.

Keywords: prey-predator system, stochastic perturbation, time-dependent delays, stability.

MSC 2010: 92B05, 92D25, 93E03.

1. INTRODUCTION

In population dynamics, the relationship between predator and prey plays an important role due to its universal existence. The increasing influence of mathematical modeling in theoretical ecology, for the last decades, the predator-prey system has been studied extensively, many important and influential results have been established in many articles and books, see [1-5].

A classical two-species predator-prey system can be expressed as follows:

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\left[r - a_{11}x(t) - a_{12}y(t)\right], \\ \dfrac{dy(t)}{dt} = y(t)\left[-d + a_{21}x(t) - a_{22}y(t)\right], \end{cases} \quad (1)$$

Where $x(t)$ and $y(t)$ stand for the population sizes of the prey and the predator, respectively, $r, d, a_{ij}(i, j = 1,2)$ are positive constants. For biological interpretation of each coefficient in system (1), we refer the reader to [3].

On the other hand, more realistic and interesting models of population interactions should take the effects of time delay into account [3, 4, 5]. In general, delay differential equations can exhibit more complicated dynamics than differential equations without delay because a time delay could cause a stable equilibrium to become unstable (see [3]).

Since all species exhibit time delays, such as their maturation time. In view of the fact that time delays are not resistant to time fluctuations, then system (1) will become

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\left[r - a_{11}x(t) - a_{12}x(t-\tau_1(t)) - a_{13}y(t-\tau_2(t))\right], \\ \dfrac{dy(t)}{dt} = y(t)\left[-d + a_{21}x(t-\tau_3(t)) - a_{22}y(t) - a_{23}y(t-\tau_4(t))\right], \end{cases} \quad (2)$$

where $a_{13} > 0, a_{23} > 0$, $\tau_i(t)$ $(i = 1,2,3,4)$ is a nonnegative, bounded, continuously differentiable function on $[0, +\infty)$ satisfying

$$\bar{\tau} = \max_{i=1,2,3,4} \sup_{t \geq 0}\{\tau_i(t)\} \geq 0$$

and $\tau_i'(t)$ $(i = 1,2,3,4)$ is continuous bounded functions on $[0, +\infty)$,where $\tau_i'(t) = \dfrac{d\tau_i(t)}{dt}$) .

It is easy to compute that if $ra_{21} - d(a_{11} + a_{12}) > 0$

System(2) has a positive equilibrium $X^* = (x^*, y^*)$,

Where

$$X^* = \frac{r(a_{22} + a_{23}) + da_{13}}{(a_{11} + a_{12})(a_{22} + a_{23}) + a_{13}a_{21}},$$

$$y^* = \frac{ra_{21} - d(a_{11} + a_{12})}{(a_{11} + a_{12})(a_{22} + a_{23}) + a_{13}a_{21}}.$$

However, in the real world, population systems are inevitably subjected to the environmental noises (see e.g.[6-8]). May [9] pointed out that due to environmental fluctuation, the birth rate, carrying capacity, competition coefficients and other parameters involved with the system exhibit random fluctuation to a greater or lesser extent (see e.g.[9, 10]).

Taking into account the effect of environmental noise, we suppose that the white noise affects r,d mainly. Thus each growth rate could be written as an average rate plus an error term, by the central limit theorem, the error term follows a normal distribution, the error term sometimes dependent on how much the current population sizes differ from the equilibrium state. I can replace each growth rate by an average rate plus an error term, $r \to r+\sigma_1(x-x^*)\dot{B}(t),\ -d \to -d+\sigma_2(y-y^*)\dot{B}(t),$

where σ_i^2 denotes the intensity of the noise and $\dot{B}_i(t)$ is a standard white noise, namely $B_i(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega,\ F,\{F_t\}_{t\ge0},P)$ with a filtration $\{F_t\}_{t\ge0}$ satisfying the usual conditions.

Then corresponding to system (2), we obtain the following stochastic autonomous predator-prey system with time-dependent delays:

$$\begin{cases} dx(t)=x(t)[r-a_{11}x(t)-a_{12}x(t-\tau_1(t))-a_{13}y(t-\tau_2(t))] \\ \qquad dt+\sigma_1 x(t)(x(t)-x^*)dB_1(t), \\ dy(t)=y(t)[-d+a_{21}x(t-\tau_3(t))-a_{22}y(t)-a_{23}y(t-\tau_4(t))] \\ \qquad dt+\sigma_2 y(t)(y(t)-y^*)dB_2(t), \end{cases} \quad (3)$$

with initial conditions

$x(\theta)=\varphi_1(\theta)>0,\ y(\theta)=\varphi_2(\theta)>0,\ \theta\in[-\bar{\tau},0]$, where $\varphi_i(\theta)$ is a continuous function on $[-\bar{\tau},0]$.

For simplicity, Here we introduce the following notations:

$$R_+^4=\{(x,y,u,v)\in R^4: x>0,y>0,u>0,v>0\}$$

$$\langle f(t)\rangle=\frac{1}{t}\int_0^t f(s)ds,$$

$$f^u=\sup_{t\ge0}(f(t)),\ f^l=\inf_{t\ge0}(f(t)).$$

Throughout this paper, the delay functions satisfy the following condition ($(\tau_i')^l\le(\tau_i')^u\le\tau'<1,$

where $\tau'=\max_{i=1,2,3,4}\sup_{t\ge0}\tau_i'(t)$.

2. GLOBAL STABILITY OF SYSTEM (3)

Since $x(t)$ and $y(t)$ in system (3) represent population sizes at time t, it should be nonnegative. Then, for further study, the first thing considered is whether the solution of system (3) has a unique global positive solution.

Lemma 2.1 For any given initial value $(\varphi_1(\theta),\varphi_2(\theta))\in C([-\bar{\tau},0],R_+^2)$, system (3) has a unique global positive solution $(x(t),y(t))$, on $t\ge-\bar{\tau}$ and the solution will remain in R_+^2 with probability 1.

Proof. Since the coefficients of system (3) are locally Lipschitz continuous, for any given initial value $(\varphi_1(\theta),\varphi_2(\theta))\in C([-\bar{\tau},0],R_+^2)$, there exists a local positive solution $(x(t),y(t))$ on $t\in[-\bar{\tau},\ \tau_e]$, where τ_e denotes the explosion time. To verify that this solution is global, i only need to prove $\tau_e=+\infty$ a.s. The proof is similar to [12] by defining

$$V_1(x,y)=\sqrt{x}-1-\ln\sqrt{x}+\sqrt{y}-1-\ln\sqrt{y}$$

$$\begin{aligned} V_2(x,y)&=\frac{1}{4(1-\tau')}\int_{t-\tau_1(t)}^t x^2(s-\tau_1(s))ds \\ &+\frac{1}{4(1-\tau')}\int_{t-\tau_3(t)}^t x^2(s-\tau_3(s))ds \\ &=\frac{1}{4(1-\tau')}\int_{t-\tau_2(t)}^t y^2(s-\tau_1(s))ds \\ &+\frac{1}{4(1-\tau')}\int_{t-\tau_4(t)}^t y^2(s-\tau_4(s))ds, \end{aligned}$$

$$V(x,y)=V_1(x,y)+V_2(x,y).$$

Applied itô's formula to $V_1(x,y)$ have

$$\begin{aligned} dV_1(x,y)=LV_1(&x,x(t-\tau_1(t),x(t-\tau_3(t), \\ &y,y(t-\tau_2(t),y(t-\tau_4(t))dt \\ &+0.5\sigma_1(\sqrt{x(t)-1})(x(t)-x^*)dB_1(t) \\ &+0.5\sigma_2(\sqrt{y(t)}-1)(y(t)-y^*)dB_2(t), \end{aligned}$$

Where

$$LV_1(x,x(t-\tau_1(t),x(t-\tau_3(t),y,y(t-\tau_2(t),y(t-\tau_4(t))dt$$

$$\begin{aligned} &=\frac{(\sqrt{x}-1)}{2}[r-a_{11}x-a_{12}x(t-\tau_1(t))-a_{13}y(t-\tau_2(t))] \\ &+\frac{(2-\sqrt{x})}{8}\sigma_1^2(x-x^*)^2+\frac{(\sqrt{y}-1)}{2} \\ &[-d+a_{21}x(t-\tau_3(t))-a_{22}y-a_{23}y(t-\tau_4(t))]+\frac{(2-\sqrt{x})}{8}\sigma_2^2(y-y^*)^2 \\ &\le\frac{(\sqrt{x}-1)}{2}[r-a_{11}x]+a_{12}\frac{(\sqrt{x}-1)}{2}x(t-\tau_1(t)) \\ &+a_{13}\frac{(\sqrt{x}-1)}{2}y(t-\tau_2(t))+\frac{(\sqrt{y}-1)}{2}[d-a_{22}y] \\ &+a_{21}\frac{(\sqrt{y}-1)}{2}x(t-\tau_3(t))+a_{23}\frac{(\sqrt{y}-1)}{2}y(t-\tau_4(t)) \\ &+\frac{(2-\sqrt{x})}{8}\sigma_1^2(x-x^*)^2+\frac{(2-\sqrt{x})}{8}\sigma_2^2(y-y^*)^2 \\ &\le\frac{(\sqrt{x}-1)}{2}[r-a_{11}x]+\frac{(\sqrt{y}-1)}{2}[d-a_{22}y] \\ &+\frac{(2-\sqrt{x})}{8}\sigma_1^2(x-x^*)^2+\frac{(2-\sqrt{y})}{8}\sigma_2^2(y-y^*)^2 \\ &+a_{12}^2\frac{(\sqrt{x}+1)^2}{4}+\frac{x^2(t-\tau_1(t))}{4}+a_{13}^2\frac{(\sqrt{x}+1)^2}{4}+\frac{y^2(t-\tau_2(t))}{4} \\ &+a_{21}^2\frac{(\sqrt{y}+1)^2}{4}+\frac{x^2(t-\tau_3(t))}{4}+a_{23}^2\frac{(\sqrt{y}+1)}{4}+\frac{y^2(t-\tau_4(t))}{4}. \end{aligned}$$

Notice that $\tau'<1$, then we have

$$\begin{aligned}LV_2(x,y) &\le \frac{1}{4(1-\tau')}x^2(t)-\frac{1-\tau_1'(t)}{4(1-\tau')}x^2(t-\tau_1(t))+\frac{1}{4(1-\tau')}x^2(t)\\&-\frac{1-\tau_3'(t)}{4(1-\tau')}x^2(t-\tau_3(t))+\frac{1}{4(1-\tau')}y^2(t)\\&-\frac{1-\tau_2'(t)}{4(1-\tau')}y^2(t-\tau_2(t))+\frac{1}{4(1-\tau')}y^2(t)-\frac{1-\tau_4'(t)}{4(1-\tau')}y^2(t-\tau_4(t))\\&\le \frac{1}{4(1-\tau')}x^2(t)-\frac{1}{4}x^2(t-\tau_1(t))+\frac{1}{4(1-\tau')}x^2(t)\\&-\frac{1}{4}x^2(t-\tau_3(t))+\frac{1}{4(1-\tau')}y^2(t)-\frac{1}{4}y^2(t-\tau_2(t))\\&+\frac{1}{4(1-\tau')}y^2(t)-\frac{1}{4}y^2(t-\tau_4(t)).\end{aligned}$$

So

$$\begin{aligned}LV(x,y) &\le \frac{(\sqrt{x}-1)}{2}[r-a_{11}x]+\frac{(\sqrt{y}-1)}{2}[d-a_{22}y]\\&+\frac{(2-\sqrt{x})}{8}\sigma_1^2(x-x^*)^2+\frac{(2-\sqrt{y})}{8}\sigma_2^2(y-y^*)^2\\&+a_{12}^2\frac{(\sqrt{x}+1)^2}{4}+a_{13}^2\frac{(\sqrt{x}+1)^2}{4}+a_{21}^2\frac{(\sqrt{y}+1)^2}{4}\\&+a_{23}^2\frac{(\sqrt{y}+1)^2}{4}+\frac{x^2}{2(1-\tau')}+\frac{y^2}{2(1-\tau')}\\&\le K,\end{aligned}$$

where K is a positive constant. By the similar proof of [12], we can obtain the desired assertion, it is omitted in here.

Theorem 2.2 Let

$$A:=a_{11}-\frac{x^*\sigma_1^2}{2}-\frac{a_{12}+a_{13}}{2}-\frac{a_{12}}{2(1-(\tau_1')^u)}-\frac{a_{21}}{2(1-(\tau_3')^u)},$$

$$B:=a_{22}-\frac{y^*\sigma_1^2}{2}-\frac{a_{21}+a_{23}}{2}-\frac{a_{13}}{2(1-(\tau_2')^u)}-\frac{a_{23}}{2(1-(\tau_4')^u)},$$

If $A>0$, $B>0$, and $ra_{21}-d(a_{11}+a_{12})>0$ hold, for any given initial value $(\varphi_1(\theta),\varphi_2(\theta))\in C([-\dot{\tau},0],R_+^2)$, there is an unique positive equilibrium state (x^*,y^*) of system (3) which is globally asymptotically stable almost surely (a.s.).

Proof. if $ra_{21}-d(a_{11}+a_{12})>0$, then, system (3) can be rewritten as

$$\begin{cases}dx(t)=x(t)[-a_{11}(x(t)-x^*)-a_{12}x(t-\tau_1(t)-x^*)-a_{13}(y(t-\tau_2(t)-y^*)]\\ \qquad dt+\sigma_1 x(t)(x(t)-x^*)dB_1(t),\\ dy(t)=y(t)[a_{21}(x(t-\tau_3(t)-x^*)-a_{22}(y(t)-y^*)-a_{23}(y(t-\tau_4(t))-y^*]\\ \qquad dt+\sigma_2 y(t)(y(t)-y^*)dB_2(t).\end{cases}$$

Define

$$V_3(x)=x-x^*-x^*\ln(x/x^*),\ V_4(y)=y-y^*-y^*\ln(y/y^*),$$

$$V_5(x)=\frac{a_{12}}{2(1-(\tau_1')^u)}\int_{t-\tau_1(t)}^t(x(s)-x^*)^2ds,$$

$$V_6(y)=\frac{a_{13}}{2(1-(\tau_2')^u)}\int_{t-\tau_2(t)}^t(y(s)-y^*)^2ds,$$

$$V_7(x)=\frac{a_{21}}{2(1-(\tau_3')^u)}\int_{t-\tau_3(t)}^t(x(s)-x^*)^2ds,$$

$$V_8(y)=\frac{a_{23}}{2(1-(\tau_4')^u)}\int_{t-\tau_4(t)}^t(y(s)-y^*)^2ds.$$

Obviously, the above six functions are nonnegative on $(x(t),y(t))\in R_+^2$, applied $it\hat{o}'s$ formula leads to

$$\begin{aligned}LV_3 &= [-a_{11}(x(t)-x^*)-a_{12}(x(t-\tau_1(t))-x^*)-a_{13}(y(t-\tau_2(t))-y^*)]\\&\quad(x(t)-x^*)+\frac{x^*\sigma_1^2}{2}(x(t)-x^*)^2\\&=-(a_{11}-\frac{x^*\sigma_1^2}{2})(x-x^*)^2-a_{12}(x-x^*)(x(t-\tau_1(t))-x^*)\\&\quad-a_{13}(x-x^*)(y(t-\tau_2(t))-y^*)\\&\le-(a_{11}-\frac{x^*\sigma_1^2}{2}-\frac{a_{12}}{2}-\frac{a_{13}}{2})(x-x^*)^2\\&+\frac{a_{12}}{2}(x(t-\tau_1(t))-x^*)^2+\frac{a_{13}}{2}(y(t-\tau_2(t))-y^*)^2\end{aligned}$$

similarly,

$$\begin{aligned}LV_4 &= [-a_{22}(y(t)-y^*)+a_{21}(x(t-\tau_3(t))-x^*)-a_{23}(y(t-\tau_4(t))-y^*)]\\&\quad(y(t)-y^*)+\frac{y^*\sigma_2^2}{2}(y(t)-y^*)^2\\&\le-(a_{22}-\frac{y^*\sigma_2^2}{2}-\frac{a_{23}}{2}-\frac{a_{21}}{2})(y(t)-y^*)^2\\&+\frac{a_{21}}{2}(x(t-\tau_3(t))-x^*)^2+\frac{a_{23}}{2}(y(t-\tau_4(t))-y^*)^2.\end{aligned}$$

then $V_5(x),V_6(y),V_7(x),V_8(y)$ are well defined. We can derive

$$\begin{aligned}LV_5(x)&\le\frac{a_{12}}{2(1-(\tau_1')^u)}(x(t)-x^*)^2-\frac{a_{12}}{2}(x(t-\tau_1(t))-x^*)^2,\\LV_6(y)&\le\frac{a_{13}}{2(1-(\tau_2')^u)}(y(t)-y^*)^2-\frac{a_{13}}{2}(y(t-\tau_2(t))-y^*)^2,\\LV_7(x)&\le\frac{a_{21}}{2(1-(\tau_3')^u)}(x(t)-x^*)^2-\frac{a_{21}}{2}(x(t-\tau_3(t))-x^*)^2,\\LV_8(y)&\le\frac{a_{23}}{2(1-(\tau_4')^u)}(y(t)-y^*)^2-\frac{a_{13}}{2}(y(t-\tau_4(t))-y^*)^2.\end{aligned}$$

Define

$$V(x,y)=V_3(x)+V_3(x)+V_5(x)+V_6(y)+V_7(x)+V_8(y),$$

Then

$$\begin{aligned}LV(x,y)&\le-(a_{11}-\frac{x^*\sigma_1^2}{2}-\frac{a_{12}+a_{13}}{2}-\frac{a_{12}}{2(1-(\tau_1')^u)}\\&-\frac{a_{21}}{2(1-(\tau_3')^u)})(x(t)-x^*)^2-(a_{22}-\frac{y^*\sigma_2^2}{2}\\&-\frac{a_{21}+a_{23}}{2}-\frac{a_{13}}{2(1-(\tau_2')^u)}-\frac{a_{23}}{2(1-(\tau_4')^u)}(y(t)-y^*)^2)\\&=-A(x-x^*)-B(y-y^*).\end{aligned}$$

Clearly, if $A>0$ and $B>0$ hold then the above inequality implies $LV(x,y)<0$ along all trajectories in the first quadrant except (x^*,y^*). Then we can easy obtain the positive equilibrium state (x^*,y^*) of system (3) is globally asymptotically stable a.s.

3. CONCLUSIONS

The predator-prey system with delays has received great attention and has been studied extensively, but for stochastic predator-prey system with time dependent delays case, there is very little, this paper, we studied a autonomous stochastic predator-prey system with time-dependent delays. Sufficient conditions for global stability of the positive equilibrium state are established, the obtained result shows that time-dependent delays have effects on the global stability of the positive equilibrium state.

REFERENCES

[1] H.I. Freedman, Deterministic Mathematical Models in Population Ecology,Marcel Dekker, New York, 1980.

[2] R.S. Cantrell, C. Cosner, On the dynamics of predator-prey models with the Beddington-DeAngelis functional response, J. Math. Anal. Appl. 257 (2001) 206-222.

[3]]Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.

[4] S.Liu, E.Beretta, D.Breda, Predator-prey model of Beddington-DeAngelis type with maturation and gestation delays, Nonlinear Anal. Real World Appl. 11 (2010) 4072-4091.

[5] S. Geritz, M. Gyllenberg, A mechanistic derivation of the DeAngelis-Beddington functional response, J. Theor. Biol. 314 (2012) 106-108.

[6] T.C. Gard, Persistence in stochastic food web models, Bull. Math. Biol. 46 (1984) 357-370.

[7] T.C. Gard, Introduction to Stochastic Differential Equations, Dekker, New York, 1988.

[8] X. Mao, S. Sabanis, E. Renshaw, Asymptotic behaviour of the stochastic Lotka-Volterra model, J. Math. Anal. Appl. 287 (2003) 141-156

[9] R.M. May, Stability and Complexity in Model Ecosystems, Princeton University Press, NJ, 2001.

[10] M. Liu, K. Wang, Q. Wu, Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle, Bull. Math. Biol. 73 (2011) 1969-2012

[11] D. Jiang, N. Shi, X. Li, Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation, J. Math. Anal. Appl. 340 (2008) 588-597.

[12] Q. Liu. The effects of time-dependent delays on global stability of stochastic Lotka-Volterra competitive model. Physica A. 420 (2015) 108-115.

AUTHOR'S BIOGRAPHY

Zepeng xiong is a postgraduate student of Hunan Agriculture University, she published one research pape.

Prof. Xiaoping Li: He is a Professor of Mathematics at Science college of Hunan Agriculture University, His total teaching experience is 25 years, He has published more than forty research papers in reputed national and international journals.

Xiangjun Dai : He is a teacher of Tongren University, He has published 4 research paper.