

# Quadratic Non-Polynomial Spline Method for Solving the Dissipative Wave Equation

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**Abstract:-** In this paper, a dissipative wave equation was solved by quadratic non-polynomial spline function at middles between grid points in space and finite difference discretization in time direction. The stability analysis is theoretically discussed using von Neumann method, the proposed method is shown to be conditionally stable. The accuracy of the proposed method is demonstrated by a numerical example. numerical results coupled with graphical representation explicitly reveal the complete reliability of the proposed algorithm.

**Keywords:-** Dissipative, Non-polynomial spline, Finite difference, Stability analysis, accuracy

## 1. INTRODUCTION

In the last few years, considerable interest was paid to using non-polynomial spline functions for approximating the solution of partial differential equations [1-3].we shall consider a numerical solution of the following dissipative wave equation [4]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2u_t u = g(x, t) \quad (1)$$

Over a region  $\Omega = [a \leq x \leq b] \times [t \geq 0]$ , with initial conditions

$$u(x, 0) = f_1(x), u_t(x, 0) = f_2(x) \quad (2)$$

And boundary conditions

$$u(a, t) = \psi_1(t), u(b, t) = \psi_2(t) \quad (3)$$

The functions  $f_1(x)$  and  $f_2(x)$  and their derivatives are continuous functions of  $x$ , also  $\psi_1(t)$ ,  $\psi_2(t)$  and their derivatives are continuous functions of  $t$ . In this paper, we develop a quadratic non-polynomial spline to get a smooth approximations for the solution of the problem in Eq. (1) subjected to conditions in Eq. (2) and Eq. (3). This paper is organized as follows: In section 2, a new method depends on the use of the non-polynomial splines is derived. In section 3, the stability analysis is theoretically discussed using Von Neumann method, for given values of specified parameters, the proposed method is shown to be conditionally stable. Finally, in section 4, a numerical example is included to illustrate the practical implementation of the proposed method.

### 1.1. Derivation of the Method

We create a grid with two mesh constants  $h$  and  $k$ , the grid points for this situation are  $(x_i, t_j)$  where  $x_i = a +$

$ih, i = 0, 1, \dots, n$  and  $t_j = jk, j = 0, 1, \dots$  with  $x_0 = a, x_n = b$  and  $h = (b - a)/n$

Where  $h$  and  $k$  are space step length, time step length, respectively. Let  $u(x_i, t_j)$  be the exact solution of the system of Eq. (1), Eq.(2) and Eq. (3) and  $S(x_i, t_j)$  be an approximation to the exact solution  $u(x_i, t_j)$  obtained by the spline function  $Q_i(x, t_j)$  passing through the points  $(x_i, S_i^j)$  and  $(x_{i+1}, S_{i+1}^j)$ . Each non-polynomial spline segment  $Q_i(x, t_j)$  has the form Ramadan et al. [5]  $Q_i(x, t_j) = a_i(t_j) \cos w(x - x_i) + b_i(t_j) \sin w(x - x_i) + c_i t_j$  (4)

Where  $i = 0, 1, \dots, n - 1, j \geq 0, x \in [x_i, x_{i+1}], a_i(t_j), b_i(t_j)$  and  $c_i(t_j)$  are constants and  $w$  is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and Eq. (4) reduces to quadratic polynomial spline function in  $[a, b]$  when  $w \rightarrow 0$ , choosing the spline function in this form will enable us to generalize other existing methods by arbitrary choices of the parameters  $\alpha$  and  $\beta$  which will be defined later at the end of this section. Thus, our quadratic non-polynomial spline is now defined by the relations:

$$(i) S(x, t_j) = Q_i(x, t_j), i = 0, 1, \dots, n - 1, j \geq 0$$

$$(ii) S(x, t_j) \in C^\infty[a, b] \quad (5)$$

The three coefficients in Eq. (4) need to be obtained in terms of

$$S_{i+1/2}^j, D_i^j \text{ and } M_{i+1/2}^j \text{ where (i) } Q_i(x_{i+1/2}, t_j) =$$

$$S_{i+1/2}^j, (ii) Q_i^{(1)}(x_i, t_j) = D_i^j,$$

$$(iii) Q_i^{(2)}(x_{i+1/2}, t_j) = M_{i+1/2}^j \quad (6)$$

We obtain via straightforward calculations from Eq.(4) and Eq. (6)

$$a_i(t_j) = -\frac{\sec \theta/2}{w^2} M_{i+1/2}^j - \frac{\tan \theta/2}{w} D_i^j$$

$$b_i(t_j) = \frac{1}{w} D_i^j, c_i(t_j) = S_{i+1/2}^j - \frac{1}{w^2} M_{i+1/2}^j \quad (7)$$

where  $\theta = wh$  and  $i = 0, 1, 2, \dots, n - 1$

Now using the continuity conditions (ii) in Eq. (5), that is the continuity of quadratic non-polynomial spline  $S(x, t_j)$  and its first derivative at the point  $(x_i^j, S_i^j)$ , where

the two  $Q_{i-1}(x, t_j)$  and  $Q_i(x, t_j)$  join, we have  
 $Q_{i-1}^{(m)}(x_i, t_j) = Q_i^{(m)}(x_i, t_j), m = 0, 1$

Using Eq. (4) and Eq. (7) yield the relations:

$$\frac{\tan \theta/2}{w} (D_i^j + D_{i-1}^j) = (S_{i+1/2}^j - S_{i-1/2}^j) + \frac{1}{w^2} M_{i+1/2}^j (1 - \sec \theta/2) + \frac{1}{w^2} M_{i-1/2}^j (-1 + \cos \theta \sec \theta/2) \quad (8)$$

$$(D_i^j - D_{i-1}^j) = \frac{2 \sin \theta/2}{w} M_{i-1/2}^j \quad (9)$$

From Eq. (8) and Eq. (9) we get the relation:

$$S_{i-3/2}^j - 2 S_{i-1/2}^j + S_{i+1/2}^j = \alpha (M_{i-3/2}^j + M_{i+1/2}^j) + \beta M_{i-1/2}^j, i = 2, 3, \dots, n-1 \quad (10)$$

Where:

$$\alpha = h^2 \left[ \frac{-1 + \sec \theta/2}{\theta^2} \right], \quad \beta = h^2 \left[ \frac{4 \sec \theta/2 (\sin \theta/2)^2 + 2(1 - \sec \theta/2)}{\theta^2} \right]$$

Remark:-

- (i) When  $\alpha = h^2/8$  and  $\beta = 6h^2/8$  then the scheme (10) is reduced to quadratic polynomial spline in [6,7].
- (ii) When  $\alpha = h^2/24$  and  $\beta = 22h^2/24$  then the scheme (10) is reduced to cubic polynomial spline in [8].

using the dissipative wave Eq. (1), we can

write  $M_{i-3/2}^j, M_{i-1/2}^j$  and  $M_{i+1/2}^j$  in the form

$$M_{i-3/2}^j = \frac{\partial^2 S_{i-3/2}^j}{\partial x^2} = \frac{\partial^2 S_{i-3/2}^j}{\partial t^2} + \delta_{i-3/2}^j S_{i-3/2}^j - g_{i-3/2}^j$$

$$M_{i-1/2}^j = \frac{\partial^2 S_{i-1/2}^j}{\partial x^2} = \frac{\partial^2 S_{i-1/2}^j}{\partial t^2} + \delta_{i-1/2}^j S_{i-1/2}^j - g_{i-1/2}^j$$

$$M_{i+1/2}^j = \frac{\partial^2 S_{i+1/2}^j}{\partial x^2} = \frac{\partial^2 S_{i+1/2}^j}{\partial t^2} + \delta_{i+1/2}^j S_{i+1/2}^j - g_{i+1/2}^j \quad (11)$$

Where  $\delta_i^j = 2 \frac{\partial S_i^j}{\partial t}$ , We use the Taylor series in the variable  $t$  about  $t_j$  to generate the following centered difference formula

$$\frac{\partial^2 S_i^j}{\partial t^2} \approx \frac{S_i^{j+1} - 2S_i^j + S_i^{j-1}}{k^2}$$

Where  $k = t_{j+1} - t_j$  and  $S_i^j = S(x_i, t_j)$ . The relations in Eq. (11) can be discretized in the form

$$M_{i-3/2}^j \approx \frac{S_{i-3/2}^{j+1} - 2S_{i-3/2}^j + S_{i-3/2}^{j-1}}{k^2} + \delta_{i-3/2}^j S_{i-3/2}^j - g_{i-3/2}^j$$

$$M_{i-1/2}^j \approx \frac{S_{i-1/2}^{j+1} - 2S_{i-1/2}^j + S_{i-1/2}^{j-1}}{k^2} + \delta_{i-1/2}^j S_{i-1/2}^j - g_{i-1/2}^j$$

$$M_{i+1/2}^j \approx \frac{S_{i+1/2}^{j+1} - 2S_{i+1/2}^j + S_{i+1/2}^{j-1}}{k^2} + \delta_{i+1/2}^j S_{i+1/2}^j - g_{i+1/2}^j \quad (12)$$

$$\text{And } \delta_i^j = 2 \frac{\partial S_i^j}{\partial t} \approx \frac{2(S_i^j - S_i^{j-1})}{k}$$

The use of Eq. (12) in Eq.(10) gives us the following system

$$\alpha (S_{i-3/2}^{j+1} + S_{i+1/2}^{j+1}) + \beta (S_{i-1/2}^{j+1}) = A S_{i-3/2}^j + B S_{i-1/2}^j + C S_{i+1/2}^j - \alpha (S_{i-3/2}^{j-1} + S_{i+1/2}^{j-1}) - \beta (S_{i-1/2}^{j-1}) + k^2 (\alpha (g_{i+1/2}^j + g_{i-3/2}^j) + \beta g_{i-1/2}^j) \quad (13)$$

for  $i = 2, 3, \dots, n-1$  and  $j \geq 1$

Where  $A = k^2 + 2\alpha - \alpha k^2 \delta_{i-3/2}^j$ ,

$$B = -2k^2 + 2\beta - \beta k^2 \delta_{i-1/2}^j$$

$$C = k^2 + 2\alpha - \alpha k^2 \delta_{i+1/2}^j$$

$$\text{and } \delta_i^j = 2 \frac{\partial S_i^j}{\partial t} \approx \frac{2(S_i^j - S_i^{j-1})}{k}$$

Eq. (13) consists of  $(n-2)$  linear algebraic equations in the  $(n)$  unknowns  $S_{i+1/2}^j, i = 0, 1, 2, \dots, n-1$ , so we need two more equations, one at each end of the range of integration for direct computation of  $S_{i+1/2}^j$ . These two equations are deduced by Taylor series and the method of undetermined coefficients. These equations are

$$2 S_0^j - 3 S_{1/2}^j + S_{3/2}^j = h^2 (w_0 M_{1/2}^j + w_1 M_{3/2}^j + w_2 M_{5/2}^j + w_3 M_{7/2}^j), i = 1 \quad (14)$$

$$2 S_n^j - 3 S_{n-1/2}^j + S_{n-3/2}^j = h^2 \left( w_0 M_{n-1/2}^j + w_1 M_{n-3/2}^j + w_2 M_{n-5/2}^j + w_3 M_{n-7/2}^j \right), i = n \quad (15)$$

Where  $w_i$ 's will be determined later to get the required order of accuracy. using Eq.(12) in Eq. (14) and Eq.(15) gives us the following equations

$$\frac{h^2}{k^2} (w_0 S_{1/2}^{j+1} + w_1 S_{3/2}^{j+1} + w_2 S_{5/2}^{j+1} + w_3 S_{7/2}^{j+1}) = (-3 + \frac{2h^2 w_0}{k^2} - h^2 w_0 \delta_{1/2}^j) S_{1/2}^j + (1 + \frac{2h^2 w_1}{k^2} - h^2 w_1 \delta_{3/2}^j) S_{3/2}^j + (\frac{2h^2 w_2}{k^2} - h^2 w_2 \delta_{5/2}^j) S_{5/2}^j + (\frac{2h^2 w_3}{k^2} - h^2 w_3 \delta_{7/2}^j) S_{7/2}^j - \frac{h^2}{k^2} (w_0 S_{1/2}^{j-1} + w_1 S_{3/2}^{j-1} + w_2 S_{5/2}^{j-1} + w_3 S_{7/2}^{j-1})$$

$$+h^2(w_0g_{1/2}^j + w_1g_{3/2}^j + w_2g_{5/2}^j + w_3g_{7/2}^j) + 2S_0^j \quad (16)$$

$$\begin{aligned} & \frac{h^2}{k^2}(w_0S_{n-1/2}^{j+1} + w_1S_{n-3/2}^{j+1} + w_2S_{n-5/2}^{j+1} + w_3S_{n-7/2}^{j+1}) \\ & = (-3 + \frac{2h^2w_0}{k^2} - h^2w_0\delta_{n-1/2}^j)S_{n-1/2}^j \\ & + (1 + \frac{2h^2w_1}{k^2} - h^2w_1\delta_{n-3/2}^j)S_{n-3/2}^j \\ & + (\frac{2h^2w_2}{k^2} - h^2w_2\delta_{n-5/2}^j)S_{n-5/2}^j \\ & + (\frac{2h^2w_3}{k^2} - h^2w_3\delta_{n-7/2}^j)S_{n-7/2}^j \\ & - \frac{h^2}{k^2}(w_0S_{n-1/2}^{j-1} + w_1S_{n-3/2}^{j-1} \\ & + w_2S_{n-5/2}^{j-1} + w_3S_{n-7/2}^{j-1}) \\ & + h^2(w_0g_{n-1/2}^j + w_1g_{n-3/2}^j + w_2g_{n-5/2}^j + w_3g_{n-7/2}^j) + \\ & \quad 2S_n^j \quad (17) \end{aligned}$$

we can determine the values of  $w_i$ 's by expanding

Eq.(14) and Eq. (15) in terms of

$u_0^j$  at  $i = 1$  and  $u_n^j$  at  $i = n$

$$\begin{aligned} t_i^j &= 2u_0^j - 3u_{1/2}^j + u_{3/2}^j \\ & - h^2(w_0D_x^2 u_{1/2}^j + w_1D_x^2 u_{3/2}^j \\ & + w_2D_x^2 u_{5/2}^j + w_3D_x^2 u_{7/2}^j), i = 1 \quad (18) \end{aligned}$$

And

$$\begin{aligned} t_i^j &= 2u_n^j - 3u_{n-1/2}^j + u_{n-3/2}^j \\ & - h^2(w_0D_x^2 u_{n-1/2}^j + w_1D_x^2 u_{n-3/2}^j + w_2D_x^2 u_{n-5/2}^j + \\ & w_3D_x^2 u_{n-7/2}^j), i=n \quad (19) \end{aligned}$$

Then the truncation error at  $i = 1, n$  as the following

$$t_i^j = \begin{bmatrix} \left(\frac{6}{8} - (w_0 + w_1 + w_2 + w_3)\right)h^2D_x^2 + \\ \left(\frac{1}{2} - \frac{(w_0+3w_1+5w_2+7w_3)}{2}\right)h^3D_x^3 + \\ \left(\frac{39}{192} - \frac{(w_0+9w_1+25w_2+49w_3)}{8}\right)h^4D_x^4 \\ + \left(\frac{1}{16} - \frac{(w_0+27w_1+125w_2+343w_3)}{48}\right)h^5D_x^5 + \\ \left(\frac{726}{46080} - \frac{(w_0+81w_1+625w_2+2401w_3)}{384}\right)h^6D_x^6 + \dots \end{bmatrix} u_i^j \quad (20)$$

to make  $t_i^j, i = 1, n$  of order  $O(h^6)$  we make the first four terms in Eq. (20) equal to zero, then we have  $(w_0, w_1, w_2, w_3) = (233/384, 63/384, -9/384, 1/384)$ , The spline solution of Eq.(1) with initial condition in Eq.(2) and boundary condition in Eq. (3) is based on the Eq. (13), Eq.(16) and Eq. (17). then we can

write the standard matrix equations for the non-polynomial spline method in the form

$$Q S_{i+1/2}^{j+1} = Q^* S_{i+1/2}^j - Q S_{i+1/2}^{j-1} + R_{i+1/2}^j \quad (21)$$

for  $i = 0, 1, \dots, n-1$  and  $j \geq 1$

Where

$$Q = \begin{bmatrix} \frac{h^2w_0}{k^2} & \frac{h^2w_1}{k^2} & \frac{h^2w_2}{k^2} & \frac{h^2w_3}{k^2} & 0 \\ \alpha & \beta & \alpha & 0 & \\ 0 & \alpha & \beta & \alpha & \\ & & & & \\ & & & 0 & \alpha & \beta & \alpha \\ & & & 0 & \frac{h^2w_3}{k^2} & \frac{h^2w_2}{k^2} & \frac{h^2w_1}{k^2} & \frac{h^2w_0}{k^2} \end{bmatrix}$$

$$Q^* = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ A_2 & B_2 & C_2 & 0 \\ 0 & A_3 & B_3 & C_3 \\ & & & \\ & & & 0 & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & q_4^* & q_3^* & q_2^* & q_1^* \end{bmatrix}$$

$$S_{i+1/2}^j = \left[ S_{1/2}^j \ S_{5/2}^j \ S_{7/2}^j \ \dots \ S_{n-1/2}^j \right]^T$$

$$q_1 = -3 + \frac{2h^2w_0}{k^2} - h^2w_0\delta_{1/2}^j$$

$$q_2 = 1 + \frac{2h^2w_1}{k^2} - h^2w_1\delta_{3/2}^j, q_3 = \frac{2h^2w_2}{k^2} - h^2w_2\delta_{5/2}^j$$

$$q_4 = \frac{2h^2w_3}{k^2} - h^2w_3\delta_{7/2}^j$$

$$q_1^* = -3 + \frac{2h^2w_0}{k^2} - h^2w_0\delta_{n-1/2}^j$$

$$q_2^* = 1 + \frac{2h^2w_1}{k^2} - h^2w_1\delta_{n-3/2}^j$$

$$q_3^* = \frac{2h^2w_2}{k^2} - h^2w_2\delta_{n-5/2}^j$$

$$q_4^* = \frac{2h^2w_3}{k^2} - h^2w_3\delta_{n-7/2}^j$$

And

$$R_{i+1/2}^j = h^2(w_0g_{1/2}^j + w_1g_{3/2}^j + w_2g_{5/2}^j + w_3g_{7/2}^j) + 2\psi_1(t_j), i = 1$$

$$k^2(\alpha(g_{i-3/2}^j + g_{i+1/2}^j) + \beta g_{i-1/2}^j), i = 2, 3, \dots, n-1$$

$$h^2(w_0g_{n-1/2}^j + w_1g_{n-3/2}^j + w_2g_{n-5/2}^j + w_3g_{n-7/2}^j) + 2\psi_2(t_j), i = n$$

Eq. (13), Eq.(16) and Eq. (17) imply that the  $(j+1)$ st time step requires values from the  $(j)$ st and  $(j-1)$ st time steps where  $j = 1, 2, \dots$ ; since values for  $j = 0$  are given by the first part in Eq. (3) which is

$$S_{i+1/2}^0 = u(x_{i+1/2}, 0) = f_1(x_{i+1/2}), i = 0, 1, \dots, n-1 \quad (22)$$

So, it is necessary to know the approximate values of  $u(x, t)$  at the nodal points of the first time level that is at  $t = t_1 = k$ . A Taylor series expansion at  $t = k$  may be written as

$$S_{i+1/2}^1 = S_{i+1/2}^0 + k \frac{\partial S_{i+1/2}^0}{\partial t} + \frac{k^2}{2} \frac{\partial^2 S_{i+1/2}^0}{\partial t^2} + O(k^3) \quad (23)$$

using the initial values from Eq.(2) we calculate the following equations

$$\begin{aligned} \frac{\partial S_{i+1/2}^0}{\partial t} &= u_t(x, 0) = f_2(x_{i+1/2}) \\ \frac{\partial^2 S_{i+1/2}^0}{\partial t^2} &= \frac{\partial^2 S_{i+1/2}^0}{\partial x^2} - 2u_t(x_{i+1/2}, 0) u(x_{i+1/2}, 0) \\ &+ g(x_{i+1/2}, 0) = \frac{d^2}{dx^2} f_2(x_{i+1/2}) \\ &- 2u_t(x_{i+1/2}, 0) u(x_{i+1/2}, 0) + g(x_{i+1/2}, 0) \end{aligned} \quad (24)$$

Substituting from Eq. (22) and Eq. (24) into Eq. (23), we get

$$S_{i+1/2}^1 = f_1(x_{i+1/2}) + k f_2(x_{i+1/2}) + \frac{k^2}{2} \left[ g(x_{i+1/2}, 0) + \frac{d^2}{dx^2} f_1(x_{i+1/2}) - 2f_1(x_{i+1/2}) f_2(x_{i+1/2}) \right], i = 0, 1, \dots, n-1 \quad (25)$$

## 2. STABILITY ANALYSIS

The stability of the spline method can be investigated according to the Von Neumann method in [9]. then taking  $\delta_{i+1}, \delta_i$  and  $\delta_{i-1}$  as a local constant  $d^*$ . We assume the solution of the difference Eq. (13) at the mesh points  $(i, j)$  can be expressed into Fourier mode in its complex exponential form as

$$S_i^j = \xi^j \exp(I\emptyset ih) \quad (26)$$

Where  $\emptyset$  is the wave number,  $I = \sqrt{-1}$ ,  $h$  is the element size and  $\xi^j$  is the amplification factor at time level  $j$ . for stability, we must have  $|\xi^j| \leq 1$  (otherwise  $\xi^j$  would grow unbounded) substituting Eq.(26) into Eq. (13), we obtain the following form

$$\xi^{j+1} \left\{ \alpha \left[ \exp\left(I\emptyset\left(i - \frac{3}{2}\right)h\right) + \exp\left(I\emptyset\left(i + \frac{1}{2}\right)h\right) \right] + \beta \exp\left(I\emptyset\left(i - \frac{1}{2}\right)h\right) \right\}$$

$$= \xi^j \left\{ \begin{aligned} & \left( k^2 + 2\alpha - \alpha k^2 d^* \right) \\ & \times \left[ \exp\left(I\emptyset\left(i - \frac{3}{2}\right)h\right) + \exp\left(I\emptyset\left(i + \frac{1}{2}\right)h\right) \right] \\ & + (-2k^2 + 2\beta - \beta k^2 d^*) \exp\left(I\emptyset\left(i - \frac{1}{2}\right)h\right) \end{aligned} \right\} - \xi^{j-1} \left\{ \begin{aligned} & \alpha \left[ \exp\left(I\emptyset\left(i - \frac{3}{2}\right)h\right) \right. \\ & \left. + \exp\left(I\emptyset\left(i + \frac{1}{2}\right)h\right) \right] \\ & + \beta \exp\left(I\emptyset\left(i - \frac{1}{2}\right)h\right) \end{aligned} \right\} \quad (27)$$

Dividing both sides of the last equation by  $\exp\left(I\emptyset\left(i - \frac{1}{2}\right)h\right)$ , then cancelling the common term

that is  $\xi^{j-1} \left\{ \alpha \left[ \exp\left(I\emptyset\left(i - \frac{3}{2}\right)h\right) + \exp\left(I\emptyset\left(i + \frac{1}{2}\right)h\right) \right] + \beta \right\}$ , we obtain

$$\xi^2 + 2\mu \xi + 1 = 0 \quad (28)$$

Where

$$\mu = \frac{2(\alpha^2 k^2 d^* - k^2) \cos \varphi + (\beta k^2 d^* + 2k^2)}{2(\alpha \cos \varphi + \beta)} - 1,$$

and  $\varphi = \emptyset h$

Or

$$\mu = \frac{k^2(1 - \cos \varphi)}{(\beta + 2\alpha \cos \varphi)} + \frac{kd^*}{2} - 1 \quad (29)$$

Eq. (28) is a quadratic in  $\xi$  and hence will have two roots, that is

$$\xi_{\pm} = -\mu \pm \sqrt{\mu^2 - 1}$$

For stability, we must have  $|\xi_{\pm}| \leq 1$ . Also from Eq.(28) we can observe that the product of the two values of  $\xi$  is clearly unity. So three cases arise.

Case 1: Both the roots are equal to unity. In that case the discriminant of the quadratic Eq.(28) is zero.

Case 2: One of the roots is greater than unity. In that case the discriminant is greater than zero. This means that stability condition, that is  $|\xi_{\pm}| \leq 1$ , is not satisfied.

In other words,  $\xi^j$  would grow in an unbounded manner.

Case 3: Discriminant is less than zero, that is:

$$\mu^2 - 1 \leq 0. \text{ Thus, for stability: } -1 \leq \mu \leq 1 \quad (30)$$

Using Eq. (29), the above inequality becomes:

$$-\frac{k^2 d^*}{2} \leq \frac{2k^2 (\sin \frac{\varphi}{2})^2}{\beta + 2\alpha - 4\alpha (\sin \frac{\varphi}{2})^2} \leq 2 - \frac{k^2 d^*}{2} \quad (31)$$

There two cases arises:

Case 1: for  $\beta = -2\alpha$ , inequality (31) becomes

$$-\frac{k^2 d^*}{2} \leq \frac{k^2}{-2\alpha} \leq 2 - \frac{k^2 d^*}{2} \quad (32)$$

The right inequality in (32) which can be written in the form:

$$\frac{k^2}{-2\alpha} \leq 2 - \frac{k^2 d^*}{2} \quad (33)$$

is satisfied for  $\alpha < 0$ ;  $k^2 \ll |\alpha|$ , and  $k^2$  small enough to make:

$$\left(2 - \frac{k^2 d^*}{2}\right) \rightarrow 2 \text{ and } 0 < \frac{k^2}{-2\alpha} \ll 1 \quad (34)$$

but the left inequality, that is  $(-d^*/2) \leq (-1/2\alpha)$  is valid for  $|\alpha|$  small enough and  $\alpha < 0$  to make  $(-1/2\alpha) > 0$ . Finally, we can say that our system is stable for  $\beta = -2\alpha$ ,  $\alpha < 0$  and  $k^2 \ll |\alpha|$ , such that  $|\alpha|$  and  $k^2$  are small enough.

Case 2: For  $\beta > 2\alpha$ ,  $\alpha > 0$ , the quantity

$\beta + 2\alpha - 4\alpha \left(\sin \frac{\varphi}{2}\right)^2$  is positive, so the right

inequality in (31) which can be written in the form:

$$2k^2 \left(\sin \frac{\varphi}{2}\right)^2 \leq \left(2 - \frac{k^2 d^*}{2}\right) (\beta + 2\alpha - 4\alpha \left(\sin \frac{\varphi}{2}\right)^2) \quad (35)$$

is satisfied for  $\alpha > 0$ ;  $\beta > 0$ ;  $\beta \gg 2\alpha$ ; and  $k^2 \ll \beta$  small enough to make  $2 - \frac{k^2 d^*}{2} \rightarrow 2$  and

$2k^2 \left(\sin \frac{\varphi}{2}\right)^2 \rightarrow 0$ , but the left inequality in (31) that

$$\text{is: } -d^* \leq \frac{4 \left(\sin \frac{\varphi}{2}\right)^2}{\beta + 2\alpha - 4\alpha \left(\sin \frac{\varphi}{2}\right)^2}$$

is valid for  $\alpha > 0$ ;  $\beta > 0$ ;  $\beta > 2\alpha$  such that  $\alpha$  and  $\beta$  are small enough and  $\left(\sin \frac{\varphi}{2}\right) \neq 0$

Finally, we can say that stability in this case requires  $\alpha > 0$ ;  $\beta > 0$  and  $\beta > 2\alpha$

such that  $\alpha$  and  $\beta$  and  $k^2 \ll \beta$  are small enough and  $\left(\sin \frac{\varphi}{2}\right) \neq 0$

## 2.1. Numerical Example

We now consider a numerical example to show that the numerical results are in good agreement with the theoretical analysis. all calculations are implemented by MATLAB 7.10.0. The accuracy of the method is measured by the error norm  $L_\infty$  defined as

$$L_\infty = \|u(x, t) - S(x, t)\|_\infty = \max_j |u_j - S_j|$$

Example

Consider the dissipative wave equation [4]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2u_t u = 2\sin^2 x \sin t \cos t \quad (36)$$

subject to the boundary conditions:  $u(0, t) = 0$

and  $u(\pi, t) = 0$ , And initial conditions:

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 0$$

The analytical solution is:  $u(x, t) = \sin x \cos t$

from the obtained numerical results in Tables 1- 4, we can conclude that applying non-polynomial splines in the solution of partial differential equations is a promising approach.

**Table 1:** The exact and numerical solution at

$$h = \frac{\pi}{50}, k = 0.001, t = 0.25, \alpha = 10^{-5} \text{ and } \beta = 0.005$$

$x$	Exact solution	Numerical solution
$0.07\pi$	0.211415560113106	0.212781989904912
$0.17\pi$	0.493342243118144	0.496403435602516
$0.27\pi$	0.726977150049252	0.731322133816448

$0.37\pi$	0.889450468385897	0.894631106006615
$0.47\pi$	0.964858177711116	0.970411644544139
$0.57\pi$	0.945818846039751	0.951279053551595
$0.67\pi$	0.834196175810620	0.839097661902968
$0.77\pi$	0.640916571706624	0.644800283101537
$0.87\pi$	0.384899588035650	0.387329859409562
$0.97\pi$	0.091205950934626	0.091795241347949

**Table 2:** The maximum absolute errors at  $h = \frac{\pi}{50}$

$$k = 0.001, \alpha = 10^{-5} \text{ and } \beta = 0.005$$

Time	0.10	0.15	0.20
Error	$9.8616 \times 10^{-4}$	$2.1532 \times 10^{-3}$	$3.7090 \times 10^{-3}$

**Table 3:** The maximum absolute errors at  $h = \frac{\pi}{50}$ ,  $k = 0.01$ ,

$$\alpha = -1 \text{ and } \beta = -2\alpha$$

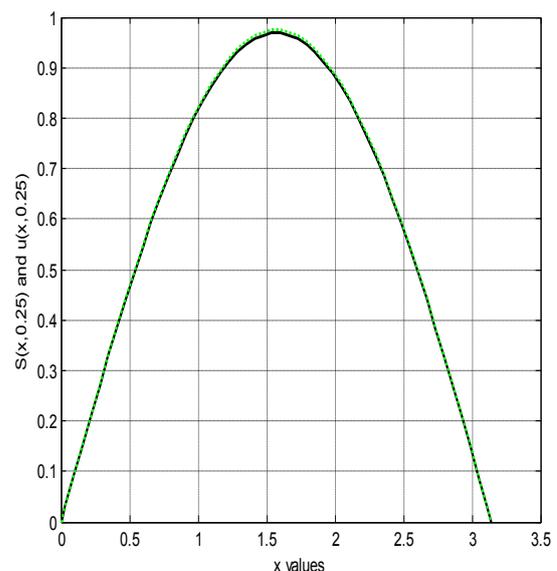
Time	1	1.5	2
Error	$2.5274 \times 10^{-3}$	$4.4852 \times 10^{-3}$	$7.5875 \times 10^{-3}$

**Table 4:** The exact and numerical solution at  $h = \frac{\pi}{50}$ ,  $k = 0.01$

$$, t = 2.5, \alpha = -1 \text{ and } \beta = -2\alpha$$

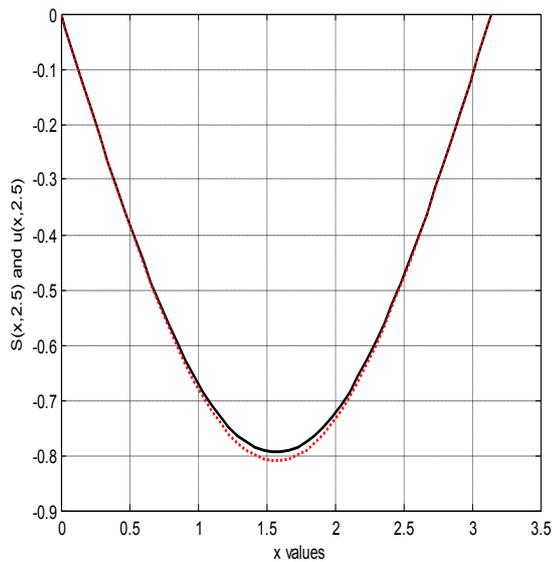
$x$	Exact solution	Numerical solution
$0.07\pi$	-0.173449822213779	-0.173880443622743
$0.17\pi$	-0.404748469382337	-0.407480685897835
$0.27\pi$	-0.596427516319343	-0.603406828812836
$0.37\pi$	-0.729724082404153	-0.741447604522964
$0.47\pi$	-0.791590171016600	-0.806147786803187
$0.57\pi$	-0.775969898356912	-0.789769859046287
$0.67\pi$	-0.684392285545858	-0.694307145831281
$0.77\pi$	-0.525821587384132	-0.530927685708137
$0.87\pi$	-0.315779808634822	-0.317328698954233
$0.97\pi$	-0.074827302049036	-0.074921135704477

While figures 1 and 2 show the exact and approximate solutions which are taking the same shape and behavior.



**Fig 1:** The exact solution (solid black) and numerical solution (dot green) at  $h = \frac{\pi}{50}$ ,  $k = 0.001$ ,

$$\alpha = 10^{-5}, \beta = 0.005 \text{ and } t = 0.25$$



**Fig 2:** The exact solution (solid black ) and numerical solution (dot red) at  $h = \frac{\pi}{50}$ ,  $k = 0.01$ ,  $\alpha = -1$ ,  $\beta = -2\alpha$  and  $t = 2.5$

### 3. CONCLUSIONS

In this paper, we have developed a new numerical method based on quadratic non-polynomial spline functions which has three coefficients in each sub interval for solving a dissipative wave equation. The method is shown to be conditional stable. The obtained numerical results showed to maintain good accuracy compared with the exact solutions. The results obtained by the proposed technique show that the approach is easy to implement and computationally very attractive .It is shown that the proposed method robust, efficient, and easy to implement for linear and nonlinear problems arising in science and engineering.

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