International Journal of Innovative Studies in Sciences and Engineering Technology (IJISSET)

ISSN 2455-4863 (Online) www.ijisset.org *Volume: 3 Issue: 1 | January 2017*

Analysis of a Stochastic Predator-Prey Model with Crowley-Martin Functional Response

Jiajia Wei, Xiaoping Li*

Science College, Hunan Agriculture University, Changsha, 410128,PR. China

**Corresponding author*

Abstract: *In this paper, we consider a density dependent predator-prey stochastic model with Crowley-Martin functional responses. For the stochastic systems, we discuss the existence of the globally positive solutions and also show that the solution of the stochastic systems will be stochastically ultimately bounded.*

Keywords: *the existence; the globally positive solutions; stochastic model; ultimate boundedness*

1. INTRODUCTION

For the last decades, many predator-prey models have been studied extensively. Many excellent results are obtained. In the predator-prey interaction, the functional response plays an important role in the population dynamics. Functional responses are of several types: HollingI-III, Ratio Dependent, Beddington-DeAngelis, Crowley-Martin, Leslie-Gower (one can see [1-6] and references cited therein).

Recently, Jai Prakash Tripathi et al. [7] introduced the following predator-prey model with Crowley-Martin Functional responses

$$\begin{cases} dx_1(t) = x_1(t)[a - bx_1(t) - \dfrac{cx_2(\mathrm{t})}{a_1 + b_1x_1(t) + c_1x_2(t) + b_1c_1x_1(t)x_2(t)}]dt \\ dx_2(t) = x_2(t)[-d - ex_2(t) + \dfrac{fx_1(\mathrm{t})}{a_1 + b_1x_1(t) + c_1x_2(t) + b_1c_1x_1(t)x_2(t)}]dt \end{cases} \quad (1.1)$$

where all parameters are positive, $x_1(t)$ and $x_2(t)$ denote respectively the densities of prey and predator species at time t. a is the intrinsic growth rate of prey species, d is the death rate of predator species, the biological meaning of all other parameters can be found in [8-10]. Then the authors discussed the existence conditions of the positive equilibrium, persistence and global stability of coexistence equilibrium.

On the other hand, in the world, population systems are often perturbed by various types of environmental noises (see e.g.[11-14]). Mao [15] pointed out that due to environmental fluctuation, the birth rate, carrying capacity, competition coefficients and other parameters involved with the system exhibit random fluctuation to a greater or lesser extent (see e.g.[16]-[22]).

Taking into account the effect of environmental noise, we assume that the environmental noise affects mainly the intrinsic growth rate a and the death rate d with

$$a \to a + \sigma_1 dB_1(t), d \to d + \sigma_2 dB_2(t),$$

where σ_i^2 denotes the intensity of the noise and $B_i(t)$ is a standard white noise, namely $B_i(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. Then corresponding to system (1.1), stochastic predator-prey model with Crowley-Martin Functional response can be expressed as follows

$$\begin{cases} dx_1(t) = x_1(t)[a - bx_1(t) - \dfrac{cx_2(\mathrm{t})}{a_1 + b_1x_1(t) + c_1x_2(t) + b_1c_1x_1(t)x_2(t)}]dt + \sigma_1 x_1(t)dB_1(t) \\ dx_2(t) = x_2(t)[-d - ex_2(t) + \dfrac{fx_1(\mathrm{t})}{a_1 + b_1x_1(t) + c_1x_2(t) + b_1c_1x_1(t)x_2(t)}]dt - \sigma_2 x_2(t)dB_2(t) \end{cases} \quad (1.2)$$

where all parameters are positive, $x_1(t)$ and $x_2(t)$ denote respectively the densities of prey and predator species at time t.

In this paper, we first investigate the existence and uniqueness of positive solution, and then, we discuss the stochastically ultimate boundedness of positive solutions.

Throughout this paper, we denote by

$R_+^2 = \{(x,y) \in R^2, x>0, y>0\}$ and assume that solution of system (1.2) satisfy the initial conditions

$x_1(0)>0$, $x_2(0)>0$. (1.3)

2. PRELIMINARIES

In this section, we introduce some lemmas which will be useful in the following.

Definition2.1 The solutions of the system (1.2) are called stochastically ultimately bounded, if for any $\varepsilon \in (0,1)$, there exist a constant $\delta > 0$ such that the solution of system (1.2) with any positive initial value has the property that

$$\lim_{t\to+\infty} \sup P\{\sqrt{x_1^2(t)+x_1^2(t)} > \delta\} < \varepsilon .$$

Lemma 3.1 [23] If $q>p>0$, then $(E|X|^q)^{\frac{1}{q}} \ge (E|X|^p)^{\frac{1}{p}}$.

Lemma 3.2 [23] (Chebyshev's inequality) For all $p>0, m>0$, we have

$$P\{|X| \ge m\} \le \frac{E(|X|^p)}{m^p}.$$

Lemma 3.2 [24] Let $x(t)$ is the solution of the n-dimensional stochastic differential Equation

$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t)$, for $t \ge t_0$.

$V(x,t)$ defined on $R^d \times [t_0,+\infty)$ such that they are continuously twice differentiable in x and t, then

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t),$$

where

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trae[g^T(x,t)V_{xx}(x,t)g(x,t)],$$

$$V_t(x,t) = \frac{\partial V}{\partial t},$$

$$V_x(x,t) = (\frac{\partial V}{\partial x_1},......,\frac{\partial V}{\partial x_d}), V_{xx}(x,t) = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}.$$

3. MAIN RESULTS

Theorem 2.1 For any given initial condition (1.3), the system (1.2) has a unique solution $x(t) = (x_1(t), x_2(t))$ on $t \in R$ and the solution will remain in R_+^2 with probability one.

Proof It is easy to see that the coefficients of system (1.2) satisfy the local Lipschitz condition. Then there exists a unique local solution $x(t) = (x_1(t), x_2(t))$ of system (1.2) on $[0,\tau_e]$, where τ_e is the explosion time, so we only need to show that $\tau_e = +\infty$ a.s.

Choosing m_0 sufficiently large such that

$$x_i(0) \in [\frac{1}{m_0}, m_0], i=1,2.$$

For each integer $m \ge m_0$, we define the stopping time

$$\tau_m = \inf\{t \in (0,\tau_e) : x_1(t) \in (1/m, m) \text{ or } x_2(t) \in (1/m, m)\}.$$

It is easy to see that τ_m is increasing as $m \to \infty$, set $\tau_\infty = \lim_{m\to\infty} \tau_m$, obviously, $\tau_\infty \le \tau_e$ a.s.. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. By reduction to absurdity, assumes that $\tau_\infty \ne \infty$, then there exist constants $T>0$ and $\varepsilon \in (0,1)$ such that

$$P\{\tau_\infty \le T\} \ge \varepsilon .$$

So, there exist an integer $m_1 \ge m_0$ such that

$P\{\tau_m \le T\} \ge \varepsilon$, for $m \ge m_1$. (2.1)

Define

$$V(x_1,x_2) = (x_1 - 1 - \ln x_1) + (x_2 - 1 - \ln x_2).$$

For $(x_1,x_2) \in R_+^2$, it is easy to verify that $V(x_1,x_2) > 0$, Applying Itô's formula to system (1.2), we have

$$dV = \frac{x_1-1}{x_1}dx_1(t) + \frac{x_2-1}{x_2}dx_2(t) + \frac{1}{2}(\sigma_1^2+\sigma_2^2)dt \quad (2.2)$$

$$\frac{x_1-1}{x_1}dx_1(t) = (ax_1 - bx_1^2 - \frac{cx_1x_2}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2})dt + \sigma_1(x_1-1)dB_1(t) \quad (2.3)$$

$$+(-a+bx_1+\frac{cx_2}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2})dt$$

$$\le [-bx_1^2 + (a+b)x_1 + (c/c_1 - a)]dt + \sigma_1(x_1-1)dB_1(t)$$

$$\frac{x_2-1}{x_2}dx_2(t)=-dx_2-ex_2^2-\frac{fx_1x_2}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2}]dt \quad (2.4)$$
$$+[d+ex_2+\frac{fx_1}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2}]dt-\sigma_2(x_2-1)dB_2(t)$$

$$\le[-ex_2^2+(e-d)x_2+(f/b_1+d)]dt-\sigma_2(x_2-1)dB_2(t).$$

Form (2.2), (2.3) and (2.4), we get

$$dV=[-bx_1^2+(a+b)x_1-ex_2^2+(e-d)x_2+(\frac{f}{b_1}+\frac{c}{c_1}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}-a+d)]dt$$
$$+\sigma_1(x_1-1)dB_1(t)-\sigma_2(x_2-1)dB_2(t).$$

It is easy to see that there exist a constant $M>0$ such that

$$-bx_1^2+(a+b)x_1-ex_2^2+(e-d)x_2+(\frac{f}{b_1}+\frac{c}{c_1}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}-a+d)\le M$$

Therefore, we have

$$dV=Mdt+\sigma_1(x_1-1)dB_1(t)-\sigma_2(x_2-1)dB_2(t) \quad (2.5)$$

Integrating both sides of the above inequality from 0 to $\tau_m\wedge T$ and then taking the expectations leads to

$$E[V(x_1(\tau_m\wedge T),x_2(\tau_m\wedge T)]\le MT+V(x_1(0),x_2(0)) \quad (2.6)$$

Let $\Omega_m=\{\tau_m\le T\},m\ge m_1$, from (2.1), we have $P(\Omega_m)\ge\varepsilon$, for $m\ge m_1$, note that for any

$\omega\in\Omega_m$, $x_1(\tau_m,\omega)$ or $x_2(\tau_m,\omega)$ equals either m or m^{-1}, and so

$$V(x_1(\tau_m,\omega),x_2(\tau_m,\omega))\ge\min(m-1-\ln m,m^{-1}-1+\ln m).$$

It follows from (2.6) that

$$V(x_1(0),x_2(0))+MT\ge E[I_{\Omega_m}(\omega)V(x_1(\tau_m,\omega),x_2(\tau_m,\omega))]$$

$$\ge\omega\min(m-1-\ln m,m^{-1}-1+\ln m),$$

where $I_{\Omega_m}(\omega)=\begin{cases}1,\omega\in\Omega_m\\0,\omega\notin\Omega_m\end{cases}$. Setting $m\to+\infty$ leads to the contradiction

$$+\infty>V(x_1(0),x_2(0))+MT=+\infty.$$

So, we have $\tau_\infty=\infty$ a.s.. The proof of Theorem 2.1 is completed.

Theorem 3.2 the solutions of the system (1.2) are stochastically ultimately bounded

Proof Applying Itô's formula to system (1.2), we have

$$d[x_1^2(t)]=2x_1(t)dx_1(t)+\sigma_1^2x_1^2(t)dt$$

$$=2x_1^2[a-bx_1-\frac{cx_2}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2}]dt+2x_1^2\sigma_1dB_1(t)$$

and

$$d[x_2^2(t)]=2x_2(t)dx_2(t)+\sigma_2^2x_2^2(t)dt$$

$$\le2x_2^2[-d-ex_2+\frac{fx_1}{a_1+b_1x_1+c_1x_2+b_1c_1x_1x_2}]dt-2x_2^2\sigma_2dB_2(t)$$

$$\le2x_2^2(f/b_1+0.5\sigma_2^2-d-ex_2)dt-2x_2^2\sigma_2dB_2(t).$$

Let $A=\max\{2a+\sigma_1^2,\frac{2f}{b_1}+\sigma_2^2-2d\}$, $B=\min\{b,e\}$, then we have

$$d[x_i^2(t)]\le Ax_i^2(t)dt-Bx_i^3(t)+(-1)^{i-1}\sigma_ix_i^2(t)dB_i(t),i=1,2. \quad (3.2)$$

Integrating both sides of (3.2) from 0 to t and then taking the expectations leads to

$$E[x_i^2(t)]\le x_i^2(0)+A\int_0^tE[x_i^2(s)]ds-B\int_0^tE[x_i^3(s)]ds. \quad (3.3)$$

From Lemma 3.1, we have

$$\{E[x_i^3(s)]\}^{1/3}\ge\{E[x_i^2(s)]\}^{1/2}. \quad (3.4)$$

Let $u(t)=E[x_i^2(t)]$, by (3.3) and (3.4), we get

$$u(t)\le u(0)+A\int_0^tu(s)ds-B\int_0^tu^{3/2}(s)]ds. \quad (3.5)$$

Considering the following integral equation

$$\begin{cases}w(t)=w(0)+A\int_0^tw(s)ds-B\int_0^tw^{3/2}(s)]ds\\ w(0)=u(0)\end{cases}. \quad (3.6)$$

That is

$$\begin{cases}\frac{dw(t)}{dt}=Aw(t)-Bw^{3/2}(t)\\ w(0)=u(0)\end{cases}. \quad (3.7)$$

It is easy to check that $w(t)=\left(\frac{CAe^{At/2}}{1+BCe^{At/2}}\right)^2$ is the solution of (3.7), where

$C = \frac{u^{1/2}(0)}{A - Bu^{1/2}(0)}$. By comparing the principle, we have

$$u(t) = E[x_i^2(t)] \le \left(\frac{CAe^{At/2}}{1 + BCe^{At/2}} \right)^2, i = 1, 2.$$

Therefore, for any $\varepsilon \in (0,1)$, let $\delta > \frac{A}{B}\sqrt{\frac{2}{\varepsilon}}$. Then by Chebyshev's inequality, we get

$$P\{\sqrt{x_1^2(t) + x_1^2(t)} \ge \delta\} \le \frac{E[x_1^2(t) + x_1^2(t)]}{\delta^2} \le \frac{2}{\delta^2} \left(\frac{CAe^{At/2}}{1 + BCe^{At/2}} \right)^2.$$

This implies

$$\lim_{t \to +\infty} \sup P\{\sqrt{x_1^2(t) + x_1^2(t)} > \delta\} \le \frac{2}{\delta^2} \left(\frac{A}{B} \right)^2 < \varepsilon$$

as required.

REFERENCES

[1] Ruan S, Xiao D. Global analysis in a predator-prey system with nonmonotonic functional response. SIAM J Appl Math 2001; 61:1445-1472.

[2] Haiyin L, Takeuchi Y. Dynamics of the density dependent predator-prey system with Bedinton-DeAngelis functional response. J Math Anal Appl 2011; 374:644-654.

[3] Upadhyay RK, Naji RK. Dynamics of three species food chain model with Crowley Martin type functional response. Chaos, Solitons and Fractals 2009; 42:1337-1346.

[4] Bereta E, Kuang Y. Global analysis in some delayed ratio dependent predator-prey systems. Nonlin Anal TMA 1998; 32: 381-408.

[5] Hsu SB, Hwang TW, Kuang Y. Global analysis of the Michaelis-Menten-type ratiodependent predator - prey system. J Math Biol 2001; 42:489-506.

[6] Liu S, Beretta E. A stage-structured predator-prey model of Bedington-DeAngelis type. SIAM Journal on Applied Mathematics 2006; 66:1101-1129.

[7] Jai Prakash Tripathi, Swati Tyagi, Syed Abbas. Global analysis of a delayed density dependent predator-prey model with Crowley-Martin functional response.Communications in Nonlinear Science and Numerical SimulationVolume 2016; 1–3:45–69

[8] Haiyin L, Takeuchi Y. Dynamics of the density dependent predator-prey system with Bedinton-DeAngelis functional response. J Math Anal Appl 2011; 374:644-654.

[9] DeAngelis DL, Goldstein RA, O'Neill RV. A model for tropic interaction. Ecology 1975; 56: 881-892.

[10] Liu S, Beretta E. A stage-structured predator-prey model of Bedington-DeAngelis type. SIAM Journal on Applied Mathematics 2006; 66:1101-1129.

[11] T.C. Gard, Persistence in stochastic food web models, Bull. Math. Biol. 46 (1984) 357-370.

[12] T.C. Gard, Introduction to Stochastic Differential Equations, Dekker, New York, 1988.

[13] Khasminskii RZ. Stochastic stability of differential equations. The Hague: Sijthoff and Noordhoff; 1980.

[14] Mao X. A note on the LaSalle-type theorems for stochastic differential delay equations. J Math Anal Appl 2002; 268: 125-42

[15] Mao X. Exponential stability of stochastic differential equations. New York: Dekker; 1994

[16] He X, Gopalsamy K. Persistence, attractivity, and delay in facultative mutualism. J Math Anal Appl 1997; 215: 154-73.

[17] Mao X. A note on the LaSalle-type theorems for

ISSN 2455-4863 (Online) www.ijisset.org *Volume: 3 Issue: 1 | January 2017*

stochastic differential delay equations. J Math Anal Appl 2002; 268:125-42.

[18] Meng.L, Ke. W. Global stability of a nonlinear stochastic predator-prey system with BeddingtonDeAngelis functional response. J. Commun Nonlinear Sci Numer Simulat 2011; 16:1114-1121.

[19] Mao X. Exponential stability of stochastic differential equations. New York: Dekker; 1994.

[20] Meng.L, Ke. W. Global asymptotic stability of a stochastic Lotka-Volterra model with infinite delays. J.Commun Nonlinear Sci Numer Simulat 2012; 17: 3115-3123.

[21] Meng.L, Ke. W. Global stability of stage-structured predator-prey models with Beddington- DeAngelis functional response. J. Commun Nonlinear Sci Numer Simulat 2011; 16:163792-3797.

[22] Mao X. Stochastic differential equations and applications. Chichester: Horwood Publishing; 1997.

[23] Wang K. Stochastic models in mathematical Biology, Beijing: Science press; 2010.

[24] Mao X. Stochastic differential equations and applications. Chichester: Horwood Publishing; 1997.