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Fixed point theorem of Kannan type s-Hilbert space

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Abstract: *The concept of s-Hilbert space is introduced as a generalization of Hilbert space, An analogue of Banach contraction principle[1] and Kannan's fixed point theorem[6] is proved in this space. One of the main directions in obtaining possible generalizations of fixed point results in Hilbert spaces is introducing new types of spaces .Our result generalizes many known results in fixed point theory.*

Keywords: *Fixed point, Hilbert space, s-Hilbert space, Cauchy sequence.*

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1. INTRODUCTION

Since the introduction of Banach contraction principle in 1922, because of its wide applications, the study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. In this paper we have introduced the concept of s-Hilbert space, which generalizes the concept of Hilbert space.

PRELIMINARIES

1.1 NORM: A norm on X is a real-valued function ||.|| : X→R defined on X such that for any x, y ∈ X and for all λ∈K

(a) ||x|| = 0 if and only if x = 0

(b) ||x+y|| ≤ ||x|| + ||y||

(c) ||λx|| = |λ| ||x||

1.2 NORMED LINEAR SPACE: It is a pair (X, ||.||) consisting of a linear space X and a norm ||.||.

We shall abbreviate normed linear space as nls.

1.3 CAUCHY SEQUENCE : A Sequence {xn} in a normed linear space X is a Cauchy sequence if for any given ε > 0, there exist n0 ∈ N such that ||xm - xn || < ε for m, n ≥ n0

1.4 CONVERGENCE CONDITION IN NLS: A sequence {xn} in a nls X is said to be Convergent to x ∈ X if for any given ε > 0, ∃ n0 ∈ N such that ||xn - x|| < ε for n ≥ n0

1.5 COMPLETENESS : A nls X is said to be complete if for every Cauchy Sequence in X converges

to an element of X.

1.6 BANACH SPACE : A Banach Space (X, ||.||) is a complete nls.

1.7 INNER PRODUCT SPACE:Let X be a linear space over the scalar field C of complex numbers.

An inner product on X is a function (. , .) : XxX → C which satisfies the following conditions

(a) (x, y) = $\overline{(y, x)}$ for x, y ∈ X

(b) (λx + μy, z) = λ (x, z) + μ (y, z) for λ, μ ∈ C, x, y, z ∈ X

(c) (x, x) ≥ 0 ; x x) = 0 iff x = 0

1.8 LAW OF PARALLELOGRAM: If x and y are any two elements of an inner product space X then

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$$

OR

$$\|x+y\|^2\le 2\|x\|^2+2\|y\|^2$$

1.9 **HILBERT SPACE** : An infinite dimensional inner product space which is complete for the norm induced by the inner product is called Hilbert Space.

1.10 s-Hilbert space : If x and y are any two elements of an inner product space X then

$$\|x+y\|^2+\|x-y\|^2=2s\|x\|^2+2s\|y\|^2$$

OR

$$\|x+y\|^2\le 2s\|x\|^2+2s\|y\|^2 \qquad \text{for } s>1$$

2. MATERIAL AND METHOD

Theorem 2.1.:Let C be a closed subset of a Hilbert space H. T is a self continuous map of C ,if x and

y are any two elements of C then Satisfying the following Kannan type condition :

$$\|Tx-Ty\|^2\le \lambda[\|x-Tx\|^2+\|y-Ty\|^2] \qquad x,y\in C,\ \lambda\in[0,\frac{1}{s+1}]$$

Then T has a unique fixed point.

3. RESULT AND DISCUSSION:

Proof of theorem: Let x_0 be an arbitrary point in C. We construct a sequence $\{x_n\}$ of points C such that: $Tx_n = x_{n+1}$, $x=x_{n-1}$, $y=x_n$ and applying the law of parallelogram,we obtain .When $x_n = x_{n+1}$ then T has a unique fixed point ,otherwise

$$Q_n = \|x_n - x_{n+1}\|^2 = \|Tx_{n-1} - Tx_n\|^2$$
$$\leq 2\lambda[\|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - Tx_n\|^2]$$
$$\leq 2\lambda[\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2]$$
$$Q_n \leq 2\lambda[Q_{n-1} + Q_n]$$
$$\Rightarrow Q_n = \frac{2\lambda}{1-2\lambda} Q_{n-1}$$
$$\Rightarrow Q_n = \beta\, Q_{n-1} \quad \text{where} \quad \beta = \frac{2\lambda}{1-2\lambda}$$
$$\Rightarrow Q_n = \beta^n Q_0 \quad \text{such that} \quad \beta < \frac{1}{s} \text{ and } \lambda < \frac{1}{s+1}$$
$$But \quad \|x_0 - Tx_0\|^2 = \|x_n - Tx_n\|^2 \quad if\ x_0 = x_n$$
$$\Rightarrow \|x_0 - x_1\|^2 = \|x_n - x_{n+1}\|^2$$
$$\Rightarrow Q_0 = Q_n$$
$$\Rightarrow Q_0 = \beta^n Q_0 \quad \text{contradiction}$$
$$\text{So} \quad Q_0 = 0$$

Hence $x_0 = x_1$

Therefore x_0 is fixed point of T .If $x_n \neq x_m$

Then

$$\|x_n - x_{n+2}\|^2 = \|Tx_{n-1} - Tx_{n+1}\|^2$$
$$\leq 2\lambda[\|x_{n-1} - Tx_{n-1}\|^2 + \|x_{n+1} - Tx_{n+1}\|^2]$$
$$\leq 2\lambda[\|x_{n-1} - x_n\|^2 + \|x_{n+1} - x_{n+2}\|^2]$$
$$\leq 2\lambda[Q_{n-1} + Q_{n+1}]$$
$$\leq 2\lambda[\beta^{n-1} Q_0 + \beta^{n+1} Q_0]$$
$$= \gamma\, \beta^{n-1} Q_0$$

So $\{x_n\}$ is a Cauchy sequence

Finally

$$\|u - Tu\|^2 \leq 2s[\|u - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_{n+1} - Tu\|^2]$$
$$= 2s[\|u - x_n\|^2 + Q_n + \|Tx_n - Tu\|^2]$$
$$\leq 2s[\|u - x_n\|^2 + Q_n + 2\lambda[\|x_n - Tx_n\|^2 + \|u - Tu\|^2]$$
$$\leq 2s[\|u - x_n\|^2 + Q_n + 2\lambda[\|x_n - x_{n+1}\|^2 + \|u - Tu\|^2]$$
$$\Rightarrow (1-2s\lambda)\|u - Tu\|^2 \leq 2s[\|u - x_n\|^2 + \beta^n Q_0 + 2\lambda[\|x_n - x_{n+1}\|^2 + \|u - Tu\|^2]$$
$$\Rightarrow (1-2s\lambda)\|u - Tu\|^2 = 0 \quad \text{as} \quad \lambda \in [0, \frac{1}{s+1}]$$
$$\Rightarrow \|u - Tu\|^2 = 0$$
$$\Rightarrow Tu = u$$
$$\Rightarrow Tx_0 = x_0$$

Therefore x_0 is fixed point

Example :

Let X=N, defined on C:XxX→X such that

$$\|x - y\|^2 = 0 \quad \text{if} \quad x = y$$

$=5\alpha$ if x ,y $\in\{1,2\}$ and $x \neq y$

$=\alpha$ if x or y $\notin\{1,2\}$ and $x \neq y$

When $s = \frac{4}{3} > 1$

Then $\|1-2\|^2 = 5\alpha > 3\alpha = \|1-3\|^2 + \|3-4\|^2 + \|4-2\|^2$

Theorem 2.2.:Let C be a closed subset of a Hilbert space H. T is a self continuous map of C ,if x and

y are any two elements of C then Satisfying the following condition :

$$\|Tx - Ty\|^2 \leq \lambda[\, a \max\{\|x - Tx\|^2, c(\|x - Ty\|^2, \|y - Tx\|^2)\}, b \max(\|x - Tx\|^2, \|y - Ty\|^2)]$$

Where $0<a<1$,$a+b<1$,$2c< \frac{4-a}{8-a}$ for all $x,y \in C$, $\lambda \in [0, \frac{1}{s+1}]$

Then T has a unique fixed point.

Proof of the theorem is easy according to theorem 2.1

4. CONCLUSION

Which shows that $\{Tx_n\}$ is Cauchy sequence ,since C is closed subspace of Hilbert space and a sequence$\{Tx_n\}$converges to same point u in T(C) .So T has fixed point.

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REFERENCES

[1] Banach, S: Sur les operations dansles ensembles abstraitsetleur application aux equations integrals. Fundam. Math. 3, 133-181 (1922)

[2] Browder, F.E.: Fixed point theorems for nonlinear semi cont ract ive mappings in Banach space, Arch, Rat, Mech, Anal , 21 , 259 -269 , (1965 - 66) .

[3] Browder, F.E. and Pet ryshyn W.V. : Cont ract ion of fixed point s of nonlinear mappings in Hilbert space, J .Math. Anl. Appl .20, 197 -228, (1967).

[4] Hichs, T. L.and Huffman, Ed. W.: Fixed point theorems of generalized Hilbert space, J. Math Anal, Appl, 64 (1978).

[5] Jungck G.: Compatible mappings and common fixed points, Internet J . Math. And Math. Sci. 9 (4) (1986), 771 -779.

[6] Kannan, R. : Some results on fixed points, Bull. Calcutta. Math. Soc., 60(1968), 71-76. 1, 1

[7] Meir, A, Keeler, E: A theorem on contraction mappings. J.Math. Anal. Appl. 28, 326-329 (1969).

[8] Sao, G.S: Common fixed point theorem for compatibility on Hilbert space, Applied Sci. Periodical , vol .9(1), Feb.2007,p.27 -29

[9] Sharma,Aradhana and Sao,G.S. :Mei r Keeler type cont ract ive condi t ions on Hilbert space, Acta Ciencia Indica vol .34(4)2008,p.1737 -1738.

[10] Sao,G.S. and Gupta S.N. :Common f ixed point theorem in Hilbert space for rational expression, Impact Jour .of Sci .and Tech.vol .4,2010,P.B. No. 1889 Lautoka Fi j I I sland,p.39 -41.

[11] Sao, G.S. and Sharma Aradhana: Generalization of Common fixed point Theorems of Naimpally and Singh in Hilbert Space, Acta Sciencia India 2008 34(4) p. 1733-34.

[12] Sharma, Aradhana and Sao, G.S. : Common Fixed Point in Banach Space International Journal of Modern Science and Engineering Technology Vol-2 Issue-8 2015 pp. 54-59.

[13] Sinha, Seema, Verma, Premlata and Sao, G.S.: Common fixed point in Hilbert space Global Jour. of engineering science and researches vol.2,issue 10(2015),pp. 86-88.

[14] Yadav, Hema, Sayyed, S.A. and Badshah, V.H:, A note on common fixed point theorem in Hilbert space, *Material Science Research India*, vol.7 (2)(2010), 515-518.