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Contraction and Fixed Point in Hilbert Space

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Abstract:- *In this paper we will prove a common fixed point theorem using contraction and rational inequality in Hilbert Space, So the purpose of this paper is establish the generalisation of contraction in Hilbert Space.*

Keywords and Phrases:- *Hilbert Space, Common Fixed Point, Parallelogram Law.*

1. INTRODUCTION

In recent years some fixed points of various type of contraction mapping in Hilbert space and Banach spaces were obtained, among others by Browder [1], Browder and Petryshyn[2],Hicks ,Huffman[3],Junck[4], Mujahid Abbas,Miko Jovanovic , Stojan Radenovic , Aleksandra Sretenovic Suzana Simic[5] and Yadav, Hema, Sayyed, S.A. and Badshah, V.H[11].

2. PRELIMINARIES

NORM: A norm on X is a real-valued function $\|.\|$: X→R defined on X such that for any x, y ∈ X and for all λ∈K

a) $\|x\|$ = 0 if and only if x = 0

b) $\|x + y\| \leq \|x\| + \|y\|$

c) $\|\lambda x\| = |\lambda| \, \|x\|$

NORMED LINEAR SPACE: It is a pair (X, $\|.\|$) consisting of a linear space X and a norm $\|.\|$ We shall abbreviate normed linear space as nls.

CAUCHY SEQUENCE: A Sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if for any given ε > 0, there exist $n_0 \in N$ such that $\|x_m - x_n\| < \varepsilon$ for m, n ≥ n_0

CONVERGENCE CONDITION IN NLS: A sequence $\{x_n\}$ in a nls X is said to be Convergent to x ∈ X if for any given ε > 0, ∃ $n_0 \in N$ such that $\|x_n - x\| < \varepsilon$ for $n \geq n_0$

COMPLETENESS: A nls X is said to be complete if for every Cauchy Sequence in X converges to an element of X.

BANACH SPACE: A Banach Space (X, $\|.\|$) is a complete nls.

INNER PRODUCT SPACE: Let X be a linear space over the scalar field C of complex numbers. An inner product on X is a function (. , .) : XxX → C which satisfies the following conditions

(a) (x, y) = $\overline{(y, x)}$ for x, y ∈ X

(b) (λx + μy, z) = λ (x, z) + μ (y, z) for λ, μ ∈ C, x, y, z ∈ X

(c) (x, x) ≥ 0 ; x x) = 0 iff x = 0

LAW OF PARALLELOGRAM: If x and y are any two elements of an inner product space X then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

or $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$

HILBERT SPACE: An infinite dimensional inner product space which is complete for the norm induced by the inner product is called Hilbert Space.

3. MATERIAL AND METHODS

3.1 Theorem: If E, F and T are mapping on H satisfying E(X)⊂ T(X), F(X)⊂ T(X),ET=TE and FT=TF , Then we have

$$\|Ex - Fy\|^2 \leq, \alpha \frac{\|Tx - Ty\|^2 \{\|Tx - Fy\|^2 + |Ty - Ex\|^2\}}{\|Tx - Ex\|^2 + \|Ty - Fy\|^2} + \beta \{\|Tx - Ex\|^2 + \|Ty - Fy\|^2\} + \gamma \{\|Ty - Fy\|^2 + \|Ty - Ex\|^2\}$$

Then T,E and F has a common fixed point when $2\alpha + 2\beta + 4\gamma < 1$.

Proof: Let $Ex_{2n} = Tx_{2n+1} = x_{2n+2}$ and $FBx_{2n+1} = T x_{2n+2} = x_{2n+3}$, take $x=x_{2n}$,$y=x_{2n+1}$ then we have

$$\|x_{2n+2} - x_{2n+3}\|^2$$

$$\leq \alpha \frac{\|x_{2n+1} - x_{2n+2}\|^2 \{\|x_{2n+1} - x_{2n+3}\|^2 + |x_{2n+2} - x_{2n+2}\|^2\}}{\|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n+2} - x_{2n+3}\|^2}$$

$$+ \beta \{\|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n+2} - x_{2n+3}\|^2\}$$

$$+ \gamma \{\|x_{2n+1} - x_{2n+3}\|^2 + \|x_{2n+2} - x_{2n+2}\|^2\}$$

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$$\leq \alpha \frac{\| x_{2n+1} - x_{2n+2} \|^2 \; 2\{ \|x_{2n+1} - x_{2n+2}\|^2 + \| x_{2n+2} - x_{2n+3} \|^2 \}}{\|x_{2n+1} - x_{2n+2}\|^2 + \| x_{2n+2} - x_{2n+3} \|^2}$$

$$+ \beta \{ \|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n+2} - x_{2n+3}\|^2 \}$$

$$+ \gamma \; 2\{ \|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n+2} - x_{2n+3}\|^2 \}$$

(By applying parallelogram law)

Hence X is a Cauchy sequence on H ,but H is a Hilbert space so X is convergent ,Then T,E and F has a convergent subsequence ,therefore they have common fixed point p say. Which be limit point.

THEOREM: If T be the self map in H satisfying following then we have

$$\|x - Tx\|^2 + \|y - Ty\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2$$

$$\leq, \frac{\| y - Ty \|^2 \{1 + | x - Tx \|^2\}}{1 + \| x - y \|^2}$$

$$+ \frac{\| x - Tx \|^2 \{1 + \| y - Ty \|^2\}}{1 + \| Tx - Ty \|^2}$$

$$+ \frac{\| x - Ty \|^2 \{1 + \| y - Tx \|^2\}}{1 + \| Tx - y \|^2}$$

$$+ \frac{\| y - Tx \|^2 \{1 + \| x - y \|^2\}}{1 + \| x - Tx \|^2}]$$

Then T has fixed point.

Proof: Suppose $x=x_{2n}$, $y=x_{2n+1}$ and $Tx_{2n} = Tx_{2n+1}$ then we have

$$\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n} - x_{2n+2}\|^2$$

$$+ \|x_{2n+1} - x_{2n+1}\|^2 \leq \|x_{2n+1} - x_{2n+2}\|^2$$

$$+ \|x_{2n} - x_{2n+1}\|^2 + \|x_{2n} - x_{2n+2}\|^2 + \|x_{2n+1} - x_{2n+1}\|^2$$

Or $\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2 + 2\|x_{2n} - x_{2n+1}\|^2$

$$+2\|x_{2n+1} - x_{2n+2}\|^2$$

$$\leq \|x_{2n+1} - x_{2n+2}\|^2 + \|x_{2n} - x_{2n+1}\|^2$$

$$+ 2\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2$$

(applying law of parallelogram)

Contradiction or absurd result.

Then $\|x_{2n+1} - x_{2n+2}\|^2 = 0$

So $\|x_{2n+1} - x_{2n+2}\| = 0$

Hence H is a Cauchy sequence ,but H is a Hilbert space so it is convergent ,Then T has a convergent subsequence ,therefore T has fixed point p say.Which be limit point.

4. RESULT AND DISCUSSION

Theorem 3.1: Above shows that $\{Tx_n\}$,$\{Ex_n\}$ and $\{Fx_n\}$ are Cauchy Sequences in H as H is a Hilbert Space then Tx_n , Ex_n Fx_n converges to some point p.Hence p is common point

Theorem 3.2: Above shows that $\{Tx_n\}$ is a Cauchy Sequence in H as H is a Hilbert Space and T is self map then Tx_n converges to some point p.

5. CONCLUSION

In this paper, we have proved the existence of a fixed point of T and contraction of T in a Hilbert Space which is unique.

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